δ-INVARIANTS OF DU VAL DEL PEZZO SURFACES OF DEGREE 1

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ABSTRACT. In this article, we compute δ-invariants of Du Val del Pezzo surfaces of degree 1.

1. INTRODUCTION

1.1. **History and Results.** It is known that a smooth Fano variety admits a Kähler–Einstein metric if and only if it is K-polystable. For del Pezzo surfaces Tian and Yau proved that a smooth del Pezzo surface is K-polystable if and only if it is not a blow up of $\mathbb{P}^2$ in one or two points (see [34, 33]). Later on, Odaka-Spotti-Sun showed which Du Val del Pezzo surfaces are K-stable in [30]. A lot of research was done for threefolds (see [3, 25, 10, 26, 20, 27, 12, 5, 4, 11, 19, 18, 9]). However, not everything is known for Fano varieties of higher dimensions and threefolds showed that often the problem can be reduced to computing δ-invariant of (possibly singular) del Pezzo surfaces (see [3, 9, 11] etc). We computed δ-invarints of Du Val del Pezzo surfaces of degree ≥ 2 in previous works [15, 16, 17]. In [3] it was proven that the δ-invariant of smooth del Pezzo surface of degree 1 is given by:

THEOREM. ([3][Lemma 2.16)] Let X be a smooth del Pezzo surface X of degree 1. Then

$$\delta(X) = \begin{cases} \dfrac{15}{7} \text{ if } |-K_X| \text{ contains a cuspidal curve,} \\ \dfrac{12}{5} \text{ if } |-K_X| \text{ does not contain cuspidal curves.} \end{cases}$$

The proof of this theorem immediately gives us a corollary:

COROLLARY. Let X be a Du Val del Pezzo surface of degree 1. Let $\mathcal{P}$ be a smooth point on X, then $\delta_{\mathcal{P}}(X) \geq \frac{15}{7}$.

This is a generalization of α-invariant computations, which were done by I. Cheltsov, D. Kosta, J, Park and J. Won in a series of papers [7, 8, 31, 32] since δ and α invariants are related as $3\alpha(X) \geq \delta(X) \geq \frac{3\alpha(X)}{2}$ in case of del Pezzo surfaces. The singularity types of Du Val del Pezzo surfaces of degree one were listed in [35]. The results of computation δ-invariants of del Pezzo surfaces with Du Val singularities in [15, 16, 17] and this article confirm the results in Odaka-Spotti-Sun [30] paper and lead to finding new K-stable examples of singular Fano threefolds. Let X be a Du Val del Pezzo surface of degree 1. Then X can be realized as the double cover $X \xrightarrow{2:1} \mathbb{P}(1,1,2)$, which is ramified along a sextic curve $R \in \mathbb{P}(1,1,2)$. In this article, we compute δ-invariants of Du Val del Pezzo surfaces of degree 1. Note that when X has $\mathbb{A}_7$ singularities δ-invariant depends on whether R is reducible or irreducible. We prove that:

MAIN THEOREM. Let X be the Du Val del Pezzo surface of degree 1. Then the δ-invariant of X is uniquely determined by the by the type of singularities on X and unique elements of $|-K_X|$ containing each of singular points which is given in the following table:

Type of singularity	$\delta(X)$
$\mathbb{A}_1$, $2\mathbb{A}_1$, $3\mathbb{A}_1$, $4\mathbb{A}_1$, $5\mathbb{A}_1$, $6\mathbb{A}_1$, $7\mathbb{A}_1$, $8\mathbb{A}_1$ all elements of $\lvert -K_X\rvert$ containing singular points are nodal	2
$\mathbb{A}_1$, $2\mathbb{A}_1$, $3\mathbb{A}_1$, $4\mathbb{A}_1$, $5\mathbb{A}_1$, $6\mathbb{A}_1$, $7\mathbb{A}_1$, $8\mathbb{A}_1$ some elements of $\lvert -K_X\rvert$ containing singular points are cuspidal	$\frac{9}{5}$
$\mathbb{A}_2$, $\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_2+2\mathbb{A}_1$, $\mathbb{A}_2+3\mathbb{A}_1$, $\mathbb{A}_2+4\mathbb{A}_1$, $2\mathbb{A}_2$, $2\mathbb{A}_2+\mathbb{A}_1$, $2\mathbb{A}_2+2\mathbb{A}_1$, $3\mathbb{A}_2$, $3\mathbb{A}_2+\mathbb{A}_1$, $4\mathbb{A}_2$ all elements of $\lvert -K_X\rvert$ containing $\mathbb{A}_2$ singular points are nodal	$\frac{12}{7}$
$\mathbb{A}_2$, $\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_2+2\mathbb{A}_1$, $\mathbb{A}_2+3\mathbb{A}_1$, $\mathbb{A}_2+4\mathbb{A}_1$, $2\mathbb{A}_2$, $2\mathbb{A}_2+\mathbb{A}_1$, $2\mathbb{A}_2+2\mathbb{A}_1$, $3\mathbb{A}_2$, $3\mathbb{A}_2+\mathbb{A}_1$, $4\mathbb{A}_2$ some elements of $\lvert -K_X\rvert$ containing $\mathbb{A}_2$ singular points are cuspidal	$\frac{3}{2}$
$\mathbb{A}_4$, $\mathbb{A}_4+\mathbb{A}_1$, $\mathbb{A}_4+2\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_2$, $\mathbb{A}_4+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_4+\mathbb{A}_3$, $2\mathbb{A}_4$	$\frac{4}{3}$
$\mathbb{A}_5$, $\mathbb{A}_5+\mathbb{A}_1$, $\mathbb{A}_5+2\mathbb{A}_1$, $\mathbb{A}_5+\mathbb{A}_2$, $\mathbb{A}_5+\mathbb{A}_2+\mathbb{A}_1$, $\mathbb{A}_5+\mathbb{A}_3$	$\frac{6}{5}$
$\mathbb{A}_6$, $\mathbb{A}_6+\mathbb{A}_1$	$\frac{9}{8}$
$\mathbb{A}_7$, $\mathbb{A}_7+\mathbb{A}_1$ and R irreducible	$\frac{18}{17}$
$\mathbb{A}_7$, $\mathbb{A}_7+\mathbb{A}_1$ and R reducible	1
$\mathbb{A}_8$, $\mathbb{D}_4$, $\mathbb{D}_4+\mathbb{A}_1$, $\mathbb{D}_4+2\mathbb{A}_1$, $\mathbb{D}_4+3\mathbb{A}_1$, $\mathbb{D}_4+4\mathbb{A}_1$,$\mathbb{D}_4+\mathbb{A}_2$, $\mathbb{D}_4+\mathbb{A}_3$, $2\mathbb{D}_4$	1
$\mathbb{D}_5$, $\mathbb{D}_5+\mathbb{A}_1$, $\mathbb{D}_5+2\mathbb{A}_1$, $\mathbb{D}_5+\mathbb{A}_2$, $\mathbb{D}_5+\mathbb{A}_3$	$\frac{6}{7}$
$\mathbb{D}_6$, $\mathbb{D}_6+\mathbb{A}_1$, $\mathbb{D}_6+2\mathbb{A}_1$	$\frac{3}{4}$
$\mathbb{D}_7$	$\frac{2}{3}$
$\mathbb{D}_8$, $\mathbb{E}_6$, $\mathbb{E}_6+\mathbb{A}_1$, $\mathbb{E}_6+\mathbb{A}_2$	$\frac{3}{5}$
$\mathbb{E}_7$, $\mathbb{E}_7+\mathbb{A}_1$	$\frac{3}{7}$
$\mathbb{E}_8$	$\frac{3}{11}$

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1.2. **Applications.** Let X be a del Pezzo surface of degree 1 with at most Du Val singularities. Let S be a weak resolution of X. We will call an image on X of a (-1)-curve in S **a line** as was done in [13]. The immediate corollaries from Main Theorem are:

Corollary 1.1. *Let X be a Du Val del Pezzo surface of degree 1 with $\mathbb{A}_n$ or $\mathbb{D}_4$ singularities then X is K-semi-stable.*

Proof. For such X have $\delta(X)\geq 1$. Thus, X is K-semi-stable by [3, Theorem 1.59]. □

Corollary 1.2 ([30])**.** *Let X be a Du Val del Pezzo surface of degree 1 with at most $\mathbb{A}_6$ singularities or a Du Val del Pezzo surface of degree 1 with $\mathbb{A}_7$ singularity and irreducible ramification divisor R then X is K-stable. Moreover, $\mathrm{Aut}(X)$ is finite.*

Proof. For such X have $\delta(X) > 1$. Thus, X is K-stable. By [6, Corollary 1.3] $\mathrm{Aut}(X)$ is finite for K-stable X. □

There are also some applications in the case of threefolds. Smooth Fano threefolds over $\mathbb{C}$ were classified in [22, 23, 28, 29] into 105 families. The detailed description of these families can be found in [3] where the problem to find all K-polystable smooth Fano threefolds in each family was posed. The output of this paper, give some alternative proofs for this problem as well as some proofs in case of singular Fano threefolds. We know ([21, 24]) that the Fano threefold $\mathbf{X}$ is K-stable if and only if for every prime divisor $\mathbf{E}$ over $\mathbf{X}$ we have

$$\beta(\mathbf{E}) = A_{\mathbf{X}}(\mathbf{E}) - S_{\mathbf{X}}(\mathbf{E}) > 0$$

where $A_{\mathbf{X}}(\mathbf{E})$ is the log discrepancy of the divisor $\mathbf{E}$ and $S_{\mathbf{X}}\big(\mathbf{E}\big) = \frac{1}{(-K_{\mathbf{X}})^3}\int\limits_0^\infty \mathrm{vol}\big(-K_{\mathbf{X}} - u\mathbf{E}\big)du$. To show this, we fix a prime divisor $\mathbf{E}$ over $\mathbf{X}$. Then we set $Z = C_{\mathbf{X}}(\mathbf{E})$. Let Q be a general point in Z. Following [2, 3] denote

$$\delta_Q\big(X, W^X_{\bullet,\bullet}\big) = \inf_{\substack{F/X \\ Q\in C_X(F)}} \frac{A_X(F)}{S\big(W^X_{\bullet,\bullet}; F\big)} \text{ and } \delta_Q\big(\mathbf{X}\big) = \inf_{\substack{\mathbf{F}/\mathbf{X} \\ Q\in C_{\mathbf{X}}(\mathbf{F})}} \frac{A_{\mathbf{X}}(\mathbf{F})}{S_{\mathbf{X}}(\mathbf{F})}$$

where the first infimum is taken by all prime divisors F over the surface X whose center on X contains Q and the second infimum is taken by all prime divisors $\mathbf{F}$ over the threefold $\mathbf{X}$ whose center on $\mathbf{X}$ contains Q.

1.2.1. *Family 1.11 (Del Pezzo Threefold of degree 1).* Let $\mathbf{V}$ be a Fano threefold with canonical Gorenstein singularities such that $-K_{\mathbf{V}} \sim 2H$ for some $H \in \mathrm{Pic}(\mathbf{V})$ with $H^3 = 1$. Then $\mathbf{V}$ is a sextic hypersurface in $\mathbb{P}(1, 1, 1, 2, 3)$ and a del Pezzo threefold of degree 1. A general element in $|H|$ is a Du Val del Pezzo surface of degree 1 and if $\mathbf{V}$ has isolated singularities then a general surface in $|H|$ is a smooth.

Remark 1.3. If $\mathbf{V}$ is smooth then $\mathbf{V}$ is a smooth Fano threefold in Family 1.11. and all smooth Fano threefolds in this family can be obtained this way. Every smooth element in this family is known to be K-stable [3].

Main Theorem gives the following corollary:

Corollary 1.4. *Suppose that for any point Q on $\mathbf{V}$ there exists an element $X \in |H|$ such that $Q \in X$ and X has at most $\mathbb{A}_2$ singularities then $\mathbf{V}$ is K-stable.*

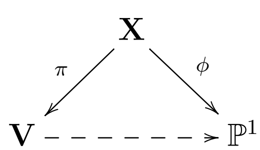
Proof. Suppose X is an irreducible element of $|H|$ then $S_{\mathbf{V}}(X) < 1$. As explained above we fix a prime divisor $\mathbf{E}$ over $\mathbf{V}$. Then we set $Z = C_{\mathbf{V}}(\mathbf{E})$ and if $\beta(\mathbf{E}) \leqslant 0$, then $\delta_Q(X, W^X_{\bullet,\bullet}) \leqslant 1$. Let Q be a general point in Z, Let X be the general element of $|H|$ that contains Q. The divisor $-K_{\mathbf{V}} - uX$ is nef if and only if $u \leq 2$ and the Zariski Decomposition is given by by $P(u) = -K_{\mathbf{V}} - uX \sim (2-u)X$ and $N(u) = 0$ for $u \in [0, 2]$. By [3, Corollary 1.110] for any divisor F such that $Q \in C_X(F)$ over X

we get:

$$S\big(W^X_{\bullet,\bullet};F\big)=\frac{3}{(-K_{\mathbf{V}})^3}\left(\int_0^\tau\big(P(u)^2\cdot X\big)\cdot\mathrm{ord}_Q\Big(N(u)\big|_X\Big)du+\int_0^\tau\int_0^\infty\mathrm{vol}\big(P(u)\big|_X-vF\big)dvdu\right)=$$
$$=\frac{3}{8}\int_0^\tau\int_0^\infty\mathrm{vol}\big(P(u)\big|_X-vF\big)dvdu=\frac{3}{8}\int_0^2(2-u)^3\int_0^\infty\mathrm{vol}\big(-K_X-wF\big)dwdu=$$
$$=\frac{3}{8}\int_0^2(2-u)^3\Big(\int_0^\infty\mathrm{vol}\big(-K_X-wF\big)dw\Big)du=\frac{3}{8}\int_0^2(2-u)^3S_X(F)du=\frac{3}{2}S_X(F)\le\frac{3}{2}\frac{A_X(F)}{\delta_Q(X)}$$

We get that $\delta_Q(\mathbf{V})\ge\frac{2}{3}\delta_Q(X)$. For X with at most $\mathbb{A}_2$-singularities we have $\delta_Q(X)\ge\frac{3}{2}$. If Q is a singular point and there exists an element X of $|H|$ with $\delta_Q(X)=\frac{3}{2}$ then $\frac{A_{\mathbf{X}}(\mathbf{E})}{S_{\mathbf{X}}(\mathbf{E})}>\min\Big\{\frac{1}{S_{\mathbf{X}}(X)},\delta_Q\big(X,W^X_{\bullet,\bullet}\big)\Big\}$ from [3, Corollary 1.108.] and otherwise we choose X with $\delta_Q(X)>\frac{3}{2}$ so $\delta_Q(\mathbf{V})>1$ if X has at most $\mathbb{A}_2$-singularities and the result follows. □

1.2.2. *Family 2.1.* Let $\mathbf{V}$ be a Fano threefold with canonical Gorenstein singularities such that $-K_{\mathbf{V}}\sim 2H$ for some $H\in\mathrm{Pic}(\mathbf{V})$ with $H^3=1$. Then $\mathbf{V}$ is a sextic hypersurface in $\mathbb{P}(1,1,1,2,3)$ and a del Pezzo threefold of degree 1. Let S_1 and S_2 be two distinct surfaces in the linear system $|H|$, and let $\mathcal{C}=S_1\cap S_2$. Suppose that the curve $\mathcal{C}$ is smooth. Then $\mathcal{C}$ is an elliptic curve by the adjunction formula. Let $\pi:\mathbf{X}\to\mathbf{V}$ be the blow up of the curve $\mathcal{C}$, and let E be the π-exceptional surface. We have the following commutative diagram:



Where $\mathbf{V}\dashrightarrow\mathbb{P}^1$ is the rational map given by the pencil that is generated by S_1 and S_2, and ϕ is a fibration into del Pezzo surfaces of degree 1.

Remark 1.5. If R is smooth then $\mathbf{X}$ is a smooth Fano threefold in Family 2.1. and all smooth Fano threefolds in this family can be obtained this way. Every smooth Fano threefold in this family is known to be K-stable [9].

Main Theorem gives the following corollary:

Corollary 1.6. *If every fiber X of ϕ at most $\mathbb{D}_4$ singularities, then $\mathbf{X}$ is K-stable.*

Proof. If X is an irreducible fiber of p_1 then we have $S_{\mathbf{X}}(X)<1$. We now fix a prime divisor $\mathbf{E}$ over $\mathbf{X}$. Then we set $Z=C_{\mathbf{X}}(\mathbf{E})$. Let Q be the point on Z. let X be the fiber of ϕ that passes through Q. Then $-K_{\mathbf{X}}-uX$ is nef if and only if $u\le 2$ and the Zariski Decomposition is given by

$$P(u)=\begin{cases}-K_{\mathbf{X}}-uX\sim(2-u)X+E \text{ if } u\in[0,1],\\ -K_{\mathbf{X}}-uX-(u-1)E\sim(2-u)\pi^*(H)\text{ if } u\in[1,2],\end{cases}\quad\text{and } N(u)=\begin{cases}0\text{ if } u\in[0,1],\\ (u-1)E\text{ if } u\in[1,2],\end{cases}$$

We apply Abban-Zhuang method to prove that $Q\notin E\cong\mathcal{C}\times\mathbb{P}^1$. By [3, Corollary 1.110] for any divisor F such that $Q\in C_X(F)$ over X we get:

$$S\big(W^{X}_{\bullet,\bullet};F\big)=\frac{3}{(-K_{\mathbf{X}})^3}\Bigg(\int_0^\tau\big(P(u)^2\cdot X\big)\cdot\mathrm{ord}_Q\Big(N(u)\big|_X\Big)du+\int_0^\tau\int_0^\infty\mathrm{vol}\big(P(u)\big|_X-vF\big)dvdu\Bigg)=$$
$$=\frac{3}{4}\int_0^\tau\int_0^\infty\mathrm{vol}\big(P(u)\big|_X-vF\big)dvdu=$$
$$=\frac{3}{4}\Bigg(\int_0^1\int_0^\infty\mathrm{vol}\big(-K_X-vF\big)dvdu+\int_1^2\int_0^\infty\mathrm{vol}\big(-K_X-(u-1)E|_X-vF\big)dvdu\Bigg)=$$
$$=\frac{3}{4}\Bigg(\int_0^\infty\mathrm{vol}\big(-K_X-vF\big)dv+\int_1^2(2-u)^3\int_0^\infty\mathrm{vol}\big(-K_X-(u-1)E|_X-vF\big)dv\Bigg)=$$
$$=\frac{3}{4}\Bigg(\int_0^\infty\mathrm{vol}\big(-K_X-vF\big)dv+\int_1^2(2-u)^3\int_0^\infty\mathrm{vol}\big(-K_X-wF\big)dwdu\Bigg)=$$
$$=\frac{3}{4}\Bigg(\int_0^\infty\mathrm{vol}\big(-K_X-vF\big)dv+\int_1^2(2-u)^3\int_0^\infty\mathrm{vol}\big(-K_X-wF\big)dwdu\Bigg)=$$
$$=\frac{3}{4}\Bigg(S_X(F)+\frac{1}{4}\cdot S_X(F)\Bigg)=\frac{15}{16}S_X(F)\leq\frac{15}{16}\cdot\frac{A_X(F)}{\delta_Q(X)}$$

We see that $\delta_Q(\mathbf{X})\geq\frac{16}{15}\delta_Q(X)$. Thus, by Main Theorem if every fiber of p_1 has at most $\mathbb{D}_4$ singularities the result follows. □

2. Proof of Main Theorem via Kento Fujita's formulas

Let X be a Du Val del Pezzo surface, and let S be a minimal resolution of X. Let $f\colon\widetilde{X}\to X$ be a birational morphism, let E be a prime divisor in $\widetilde{X}$. We say that E is a prime divisor *over* X. If E is f-exceptional, we say that E is an exceptional invariant prime divisor *over* X. We will denote the subvariety $f(E)$ by $C_X(E)$. Let

$$S_X(E)=\frac{1}{(-K_X)^2}\int_0^\tau\mathrm{vol}(f^*(-K_X)-vE)dv\text{ and }A_X(E)=1+\mathrm{ord}_E(K_{\widetilde{X}}-f^*(K_X)),$$

where $\tau=\tau(E)$ is the pseudo-effective threshold of E with respect to $-K_X$. Let Q be a point in X. We can define a local δ-invariant and a global δ-invariant now

$$\delta_Q(X)=\inf_{\substack{E/X\\ Q\in C_X(E)}}\frac{A_X(E)}{S_X(E)}\text{ and }\delta(X)=\inf_{Q\in X}\delta_Q(X)$$

where the infimum runs over all prime divisors E over the surface X such that $Q\in C_X(E)$. Similarly, for the surface S and a point P on S we define:

$$\delta_P(S)=\inf_{\substack{F/S\\ P\in C_S(F)}}\frac{A_S(F)}{S_S(F)}\text{ and }\delta(S)=\inf_{P\in S}\delta_P(S)$$

where $S_S(F)$ and $A_S(F)$ are defined as $S_X(E)$ and $A_X(E)$ above. Note that it is clear that

$$\delta(X)=\delta(S)\text{ and }\delta_Q(X)=\inf_{P:Q=f(P)}\delta_P(S)$$

Several results can help us to estimate δ-invariants. Let C be a smooth curve on S containing P. Set

$$\tau(C)=\sup\Big\{v\in\mathbb{R}_{\geqslant 0}\ \Big|\ \text{the divisor } -K_S-vC \text{ is pseudo-effective}\Big\}.$$

For $v\in[0,\tau]$, let $P(v)$ be the positive part of the Zariski decomposition of the divisor $-K_S-vC$, and let $N(v)$ be its negative part. Then we set

$$S\big(W^C_{\bullet,\bullet};P\big)=\frac{2}{K_S^2}\int_0^{\tau(C)}h(v)dv,\ \text{where } h(v)=\big(P(v)\cdot C\big)\times\big(N(v)\cdot C\big)_P+\frac{\big(P(v)\cdot C\big)^2}{2}.$$

It follows from [3, Theorem 1.7.1] that:

$$\delta_P(S)\geqslant\min\left\{\frac{1}{S_S(C)},\frac{1}{S(W^C_{\bullet,\bullet},P)}\right\}. \tag{2.1}$$

Unfortunately, using this approach we do not always get a good estimation. In this case, we can try to apply the generalization of this method. Let $\sigma\colon\widehat{S}\to S$ be a weighted blowup of the point P on S. Suppose, in addition, that $\widehat{S}$ is a Mori Dream space Then

- the σ-exceptional curve E_P such that $\sigma(E_P)=P$, it is smooth and isomorphic to $\mathbb{P}^1$,
- the log pair $(\widehat{S},E_P)$ has purely log terminal singularities.

Thus, the birational map σ a plt blowup of a point P. Write

$$K_{E_P}+\Delta_{E_P}=\big(K_{\widehat{S}}+E_P\big)\big|_{E_P},$$

where Δ_{E_P} is an effective $\mathbb{Q}$-divisor on E_P known as the different of the log pair $(\widehat{S},E_P)$. Note that the log pair (E_P,Δ_{E_P}) has at most Kawamata log terminal singularities, and the divisor $-(K_{E_P}+\Delta_{E_P})$ is $\sigma|_{E_P}$-ample.
Let O be a point on E_P. Set

$$\tau(E_P)=\sup\Big\{v\in\mathbb{R}_{\geqslant 0}\ \Big|\ \text{the divisor } \sigma^*(-K_S)-vE_P \text{ is pseudo-effective}\Big\}.$$

For $v\in[0,\tau]$, let $\widehat{P}(v)$ be the positive part of the Zariski decomposition of the divisor $\sigma^*(-K_S)-vE_P$, and let $\widehat{N}(v)$ be its negative part. Then we set

$$S\big(W^{E_P}_{\bullet,\bullet};O\big)=\frac{2}{K_{\widehat{S}}^2}\int_0^{\tau(E_P)}\widehat{h}(v)dv,\ \text{where } \widehat{h}(v)=\big(\widehat{P}(v)\cdot E_P\big)\times\big(\widehat{N}(v)\cdot E_P\big)_O+\frac{\big(\widehat{P}(v)\cdot E_P\big)^2}{2}.$$

Let $A_{E_P,\Delta_{E_P}}(O)=1-\mathrm{ord}_{\Delta_{E_P}}(O)$. It follows from [3, Theorem 1.7.9] and [3, Corollary 1.7.12] that

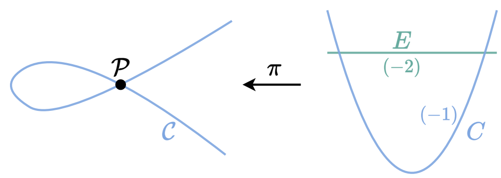
$$\delta_P(S)\geqslant\min\left\{\frac{A_S(E_P)}{S_S(E_P)},\inf_{O\in E_P}\frac{A_{E_P,\Delta_{E_P}}(O)}{S\big(W^{E_P}_{\bullet,\bullet};O\big)}\right\}, \tag{2.2}$$

where the infimum is taken over all points $O\in E_P$.
We will apply 2.1 and 2.2 to all minimal resolutions S such that $K_S^2=1$ in order to prove Main Theorem. In case X is smooth we have $S=X$. Small circles correspond to (-1)-curves and large circles correspond to (-2)-curves on dual graphs.

3. $\mathbb{A}_1$ singularity on Du Val Del Pezzo surfaces of degree 1 such that $\mathcal{C}$ is nodal

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_1$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$ and it has a node in $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X) = 2$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E is the exceptional divisor. We have $-K_S \sim C+E$. Let P be a point on S.



Suppose $P \in E$. The Zariski decomposition of the divisor $-K_S - vE \sim C + (1-v)E$ is given by:

$$P(v) = \begin{cases} -K_S - vE \text{ if } v \in \left[0, \frac{1}{2}\right] \\ -K_S - vE - (2v-1)C \text{ if } v \in \left[\frac{1}{2}, 1\right] \end{cases} \quad \text{and } N(v) = \begin{cases} 0 \text{ if } v \in \left[0, \frac{1}{2}\right] \\ (2v-1)C \text{ if } v \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - 2v^2 \text{ if } v \in \left[0, \frac{1}{2}\right] \\ 2(v-1)^2 \text{ if } v \in \left[\frac{1}{2}, 1\right] \end{cases} \quad \text{and } P(v) \cdot E = \begin{cases} 2v \text{ if } v \in \left[0, \frac{1}{2}\right] \\ 2(1-v) \text{ if } v \in \left[\frac{1}{2}, 1\right] \end{cases}$$

We have $S_S(E) = \frac{1}{2}$. Thus, $\delta_P(S) \leq 2$ for $P \in E$. Moreover, if $P \in E$:

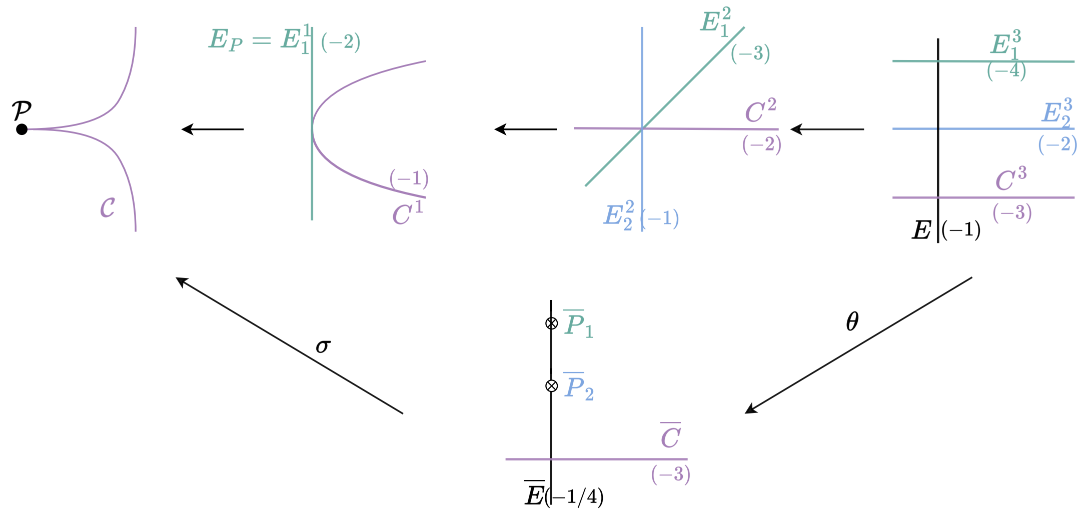
$$h(v) = \begin{cases} 2v^2 \text{ if } v \in \left[0, \frac{1}{2}\right] \\ 2v(1-v) \text{ if } v \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Thus, $S(W^E_{\bullet,\bullet}; P) \leq \frac{1}{2}$ and We get $\delta_P(S) = 2$ for $P \in E$. Which gives us $\delta_{\mathcal{P}}(X) = 2$. □

4. $\mathbb{A}_1$ singularity on Du Val Del Pezzo surfaces of degree 1 such that $\mathcal{C}$ is cuspidal

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_1$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$ and it has a cusp in $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X) = \frac{9}{5}$.

Proof. Consider the blowup $\pi_1 : S_1 \to X$ of X at $\mathcal{P}$ with the exceptional divisor E_1^1 and C^1 is a strict transform of $\mathcal{C}$. Let $\pi_2 \colon S_2 \to S_1$ be the blow up of the point $C^1 \cap E_1^1$ with the exceptional divisor E_2^2 and E_1^2, C^2 are a strict transforms of E_1^1, C^1 respectively. Let $\pi_3 \colon S_3 \to S_2$ be the blow up of the point $C^2 \cap E_1^2 \cap E_2^2$ with the exceptional divisor E and E_1^3, E_2^3, C^3 are a strict transforms of E_1^2, E_2^2, C^2 respectively. Then $(\pi_1 \circ \pi_2 \circ \pi_3)^*(-K_X) \sim C^3 + E_1^3 + 2E_2^3 + 4E$. Let $\theta \colon S_3 \to \overline{S}$ be the contraction of the curves E_1^3 and E_2^3, let $\overline{C} = \theta(C^3)$ and $\overline{E} = \theta(E)$.



Then $\overline{P}_2 = \theta(E_2^3)$ is a quotient singular point of type $\frac{1}{2}(1,1)$ and $\overline{P}_1 = \theta(E_1^3)$ is a quotient singular point of type $\frac{1}{4}(1,1)$ and the intersections are given by:

	$\overline{C}$	$\overline{E}$
$\overline{C}$	-3	1
$\overline{E}$	1	$-\frac{1}{4}$

Observe that $-K_{\overline{S}}$ is big. The Zariski decomposition of the divisor $\sigma^*(-K_X) - v\overline{E} \sim (4-v)\overline{E} + \overline{C}$ is given by

$$P(v) = \begin{cases} (4-v)\overline{E} + \overline{C} \text{ if } v \in [0,1] \\ (4-v)\overline{E} + \frac{4-v}{3}\overline{C} \text{ if } v \in [1,4] \end{cases} \quad \text{and } N(v) = \begin{cases} 0 \text{ if } v \in [0,1] \\ \frac{v-1}{3}\overline{C} \text{ if } v \in [1,4] \end{cases}$$

Moreover

$$P(v)^2 = \begin{cases} \frac{(2-v)(2+v)}{4} \text{ if } v \in [0,1] \\ \frac{(4-v)^2}{12} \text{ if } v \in [1,4] \end{cases} \quad \text{and } P(v)\cdot\overline{E} = \begin{cases} \frac{v}{4} \text{ if } v \in [0,1] \\ \frac{4-v}{12} \text{ if } v \in [1,4] \end{cases}$$

So we have $S_S(\overline{E}) = \frac{5}{3}$ for $P \in \overline{E}$. Thus, Thus, $\delta_P(S) \leq \frac{9}{5}$. Moreover, if $P \in \overline{E}\backslash\overline{C}$ or $P \in \overline{E} \cap \overline{C}$ then

$$h(v) = \begin{cases} \frac{v^2}{32} \text{ if } v \in [0,1] \\ \frac{(4-v)^2}{288} \text{ if } v \in [1,4] \end{cases} \quad \text{or } h(v) = \begin{cases} \frac{v^2}{32} \text{ if } v \in [0,1] \\ \frac{(4-v)(7v-4)}{288} \text{ if } v \in [1,4] \end{cases}$$

So $S(W^{\overline{E}}_{\bullet,\bullet}; O) = \frac{1}{12}$ or $S(W^{\overline{E}}_{\bullet,\bullet}; O) = \frac{1}{3}$. On the other hand:

$$\delta_P(S) \geqslant \min\left\{\frac{9}{5}, \inf_{O\in\overline{E}} \frac{A_{\overline{E},\Delta_{\overline{E}}}(O)}{S\big(W^{\overline{E}}_{\bullet,\bullet}; O\big)}\right\},$$

where $\Delta_{\overline{E}} = \frac{1}{2}P_1 + \frac{2}{3}P_2$. So we have

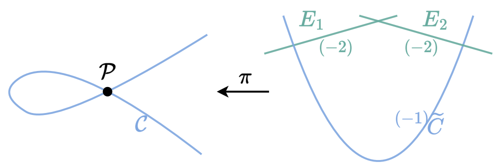
$$\frac{A_{\overline{E},\Delta_{\overline{E}}}(O)}{S(W^{\overline{E}}_{\bullet,\bullet}; O)} = \begin{cases} 3 \text{ if } O = \overline{E}\cap\overline{C}, \\ 3 \text{ if } O = P_1, \\ 4 \text{ if } O = P_2, \\ 12 \text{ otherwise.} \end{cases}$$

Thus, $\delta_{\mathcal{P}}(X) = \frac{9}{5}$. □

5. $\mathbb{A}_2$ singularity on Du Val Del Pezzo surfaces of degree 1 such that $\mathcal{C}$ is nodal

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_2$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$ and it has a node in $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X) = \frac{12}{7}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E_1 and E_2 are the exceptional divisors. We have $-K_S \sim C + E_1 + E_2$. Let P be a point on S.



Step 1. Suppose $P \in E_1 \cup E_2$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. The Zariski decomposition of the divisor $-K_S - vE_1 \sim C + (1-v)E_1 + E_2$ is given by:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{2}E_2 \text{ if } v \in \left[0, \frac{2}{3}\right] \\ -K_S - vE_1 - (2v-1)E_2 - (3v-2)C \text{ if } v \in \left[\frac{2}{3}, 1\right] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}E_2 \text{ if } v \in \left[0, \frac{2}{3}\right] \\ (2v-1)E_2 + (3v-2)C \text{ if } v \in \left[\frac{2}{3}, 1\right] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{3v^2}{2} \text{ if } v \in \left[0, \frac{2}{3}\right] \\ 3(v-1)^2 \text{ if } v \in \left[\frac{2}{3}, 1\right] \end{cases} \quad \text{and } P(v) \cdot E = \begin{cases} \frac{3v}{2} \text{ if } v \in \left[0, \frac{2}{3}\right] \\ 3(1-v) \text{ if } v \in \left[\frac{2}{3}, 1\right] \end{cases}$$

We have $S_S(E_1) = \frac{5}{9}$. Thus, $\delta_P(S) \le \frac{9}{5}$ for $P \in E_1 \backslash E_2$. Moreover, for such points we have

$$h(v) = \begin{cases} \frac{9v^2}{8} \text{ if } v \in \left[0, \frac{2}{3}\right] \\ \frac{3(1-v)(v+1)}{2} \text{ if } v \in \left[\frac{2}{3}, 1\right] \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet}; P) \le \frac{14}{27} < \frac{5}{9}$. We get $\delta_P(S) = \frac{9}{5}$ for $P \in (E_1 \cup E_2) \backslash (E_1 \cap E_2)$.

Step 2. Suppose $P = E_1 \cap E_2$. Consider the blowup $\sigma : \widetilde{S} \to S$ of S at P with the exceptional divisor E_P. Suppose $\widetilde{E}_1$, $\widetilde{E}_2$ and $\widetilde{C}$ are strict transforma of E_1, E_2 and C on S. The Zariski decomposition of the divisor $\sigma^*(-K_S) - vE_P \sim \widetilde{C} + \widetilde{E}_1 + \widetilde{E}_2 + (2-v)E_P$ is given by:

$$P(v) = \begin{cases} \sigma^*(-K_S) - vE_P - \frac{v}{3}(\widetilde{E}_1 + \widetilde{E}_2) \text{ if } v \in \left[0, \frac{3}{2}\right] \\ \sigma^*(-K_S) - vE_P - (v-1)(\widetilde{E}_1 + \widetilde{E}_2) - (2v-3)\widetilde{C} \text{ if } v \in \left[\frac{3}{2}, 2\right] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(\widetilde{E}_1 + \widetilde{E}_2) \text{ if } v \in \left[0, \frac{3}{2}\right] \\ (v-1)(\widetilde{E}_1 + \widetilde{E}_2) + (2v-3)\widetilde{C} \text{ if } v \in \left[\frac{3}{2}, 2\right]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{3} \text{ if } v \in \left[0, \frac{3}{2}\right] \\ (2-v)^2 \text{ if } v \in \left[\frac{3}{2}, 2\right] \end{cases} \quad \text{and } P(v) \cdot E_P = \begin{cases} \frac{v}{3} \text{ if } v \in \left[0, \frac{3}{2}\right] \\ 2-v \text{ if } v \in \left[\frac{3}{2}, 2\right] \end{cases}$$

We have $S_S(E_P) = \frac{7}{6}$. Thus, $\delta_P(S) \le \frac{2}{7/6} = \frac{12}{7}$ for $P = E_1 \cap E_2$. Moreover,

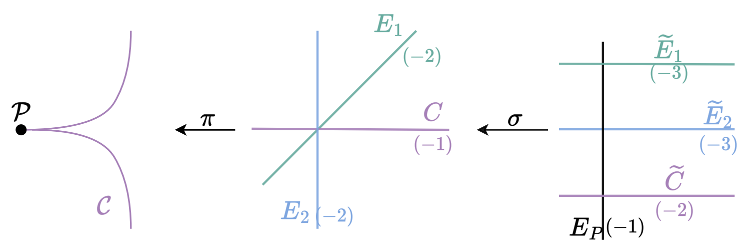
$$h(v) \le \begin{cases} \frac{v^2}{6} \text{ if } v \in \left[0, \frac{3}{2}\right] \\ \frac{(2-v)v}{2} \text{ if } v \in \left[\frac{3}{2}, 2\right] \end{cases}$$

Thus, $S(W^{E_P}_{\bullet,\bullet}; O) \le \frac{7}{12}$. We get $\delta_P(S) = \frac{12}{7}$ for $P = E_1 \cap E_2$. Thus, $\delta_{\mathcal{P}}(X) = \frac{12}{7}$. □

6. $\mathbb{A}_2$ singularity on Du Val Del Pezzo surfaces of degree 1 such that $\mathcal{C}$ is caspidal

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_2$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$ and it has a cusp in $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X) = \frac{3}{2}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. We have $-K_S \sim C + E_1 + E_2$. Let P be a point on S. Let also $\sigma : \widetilde{S} \to S$ be the blowup of a point $P = E_1 \cap E_2 \cap C$. Let $\widetilde{C}$, $\widetilde{E}_1$ and $\widetilde{E}_2$ be strict transforms of C, E_1 and E_2 on $\widetilde{S}$.



Step 1. Suppose $P \in E_1 \cup E_2$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. The Zariski decomposition of the divisor $-K_S - vE_1 \sim C + (1-v)E_1 + E_2$ is given by:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{2}E_2 \text{ if } v \in \left[0, \frac{2}{3}\right] \\ -K_S - vE_1 - (2v-1)E_2 - (3v-2)C \text{ if } v \in \left[\frac{2}{3}, 1\right] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}E_2 \text{ if } v \in \left[0, \frac{2}{3}\right] \\ (2v-1)E_2 + (3v-2)C \text{ if } v \in \left[\frac{2}{3}, 1\right] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{3v^2}{2} \text{ if } v \in \left[0, \frac{2}{3}\right] \\ 3(v-1)^2 \text{ if } v \in \left[\frac{2}{3}, 1\right] \end{cases} \quad \text{and } P(v) \cdot E = \begin{cases} \frac{3v}{2} \text{ if } v \in \left[0, \frac{2}{3}\right] \\ 3(1-v) \text{ if } v \in \left[\frac{2}{3}, 1\right] \end{cases}$$

We have $S_S(E_1) = \frac{5}{9}$. Thus, $\delta_P(S) \le \frac{9}{5}$ for $P \in E_1 \backslash E_2$. Moreover, for such points we have

$$h(v) = \begin{cases} \frac{9v^2}{8} \text{ if } v \in \left[0, \frac{2}{3}\right] \\ \frac{3(1-v)(v+1)}{2} \text{ if } v \in \left[\frac{2}{3}, 1\right] \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet}; P) \le \frac{14}{27} < \frac{5}{9}$. We get $\delta_P(S) = \frac{9}{5}$ for $P \in (E_1 \cup E_2) \backslash (E_1 \cap E_2)$.

Step 2. Suppose $P = E_1 \cap E_2$. Consider the blowup $\sigma : \widetilde{S} \to S$ of S at P with the exceptional divisor E_P. Suppose $\widetilde{E}_1$, $\widetilde{E}_2$ and $\widetilde{C}$ are strict transforma of E_1, E_2 and C on S. The Zariski decomposition of the divisor $\sigma^*(-K_S) - vE_P \sim \widetilde{C} + \widetilde{E}_1 + \widetilde{E}_2 + (3-v)E_P$ is given by:

$$P(v) = \begin{cases} \sigma^*(-K_S) - vE_P - \frac{v}{3}(\widetilde{E}_1 + \widetilde{E}_2) \text{ if } v \in \big[0,1\big] \\ \sigma^*(-K_S) - vE_P - (v-1)(\widetilde{E}_1 + \widetilde{E}_2) - \frac{v-1}{2}\widetilde{C} \text{ if } v \in \big[1,3\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(\widetilde{E}_1 + \widetilde{E}_2) \text{ if } v \in \big[0,1\big] \\ (v-1)(\widetilde{E}_1 + \widetilde{E}_2) + \frac{v-1}{2}\widetilde{C} \text{ if } v \in \big[1,3\big]. \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{3} \text{ if } v \in \big[0,1\big] \\ \frac{(3-v)^2}{6} \text{ if } v \in \big[1,3\big] \end{cases} \quad \text{and } P(v) \cdot E_P = \begin{cases} \frac{v}{3} \text{ if } v \in \big[0,1\big] \\ \frac{3-v}{6} \text{ if } v \in \big[1,3\big] \end{cases}$$

We have $S_S(E_P) = \frac{4}{3}$. Thus, $\delta_P(S) \leq \frac{2}{4/3} = \frac{3}{2}$ for $P = E_1 \cap E_2 \cap C$. Moreover, if $O \in E_P \backslash (\widetilde{E}_1 \cup \widetilde{E}_2)$ if $O \in E_P \backslash \widetilde{C}$ we have:

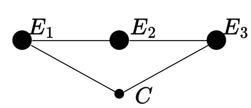
$$h(v) \leq \begin{cases} \frac{v^2}{18} \text{ if } v \in \big[0,1\big] \\ \frac{(3-v)(5v-3)}{72} \text{ if } v \in \big[1,3\big] \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{v^2}{6} \text{ if } v \in \big[0,1\big] \\ \frac{(3-v)(v+1)}{24} \text{ if } v \in \big[1,3\big] \end{cases}$$

Thus, $S(W^{E_P}_{\bullet,\bullet}; O) \leq \frac{1}{3} < \frac{2}{3}$ or $S(W^{E_P}_{\bullet,\bullet}; O) \leq \frac{5}{9} < \frac{2}{3}$. We get $\delta_P(S) = \frac{3}{2}$ for $P = E_1 \cap E_2$. Thus, $\delta_{\mathcal{P}}(X) = \frac{3}{2}$. □

7. $\mathbb{A}_3$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_3$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X) = \frac{3}{2}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E_1, E_2 and E_3 are the exceptional divisors with the following intersection:



We have $-K_S \sim C + E_1 + E_2 + E_3$. Let P be a point on S.

Step 1. Suppose $P \in E_2$. The Zariski decomposition of the divisor $-K_S - vE_2 \sim C + E_1 + (1-v)E_2 + E_3$ is given by:

$$P(v) = -K_S - vE_2 - \frac{v}{2}(E_1 + E_3) \text{ and } N(v) = \frac{v}{2}(E_1 + E_3) \text{ if } v \in \big[0,1\big]$$

Moreover,

$$(P(v))^2 = (1-v)(1+v) \text{ and } P(v) \cdot E_2 = v \text{ if } v \in \big[0,1\big]$$

We have $S_S(E_2) = \frac{2}{3}$. Thus, $\delta_P(S) \leq \frac{3}{2}$ for $P \in E_2$. Moreover, for such points we have

$$h(v) \leq v^2 \text{ if } v \in \big[0,1\big]$$

Thus, $S(W^{E_2}_{\bullet,\bullet}; P) \leq \frac{2}{3}$. We get $\delta_P(S) = \frac{3}{2}$ for $P \in E_2$.

Step 2. Suppose $P \in E_1 \cup E_3$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. The Zariski decomposition of the divisor $-K_S - vE_1 \sim C + (1-v)E_1 + E_2 + E_3$ is given by:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{3}(2E_2 + E_3) \text{ if } v \in \left[0, \frac{3}{4}\right] \\ -K_S - vE_1 - (2v-1)E_2 - (3v-2)E_3 - (4v-3)C \text{ if } v \in \left[\frac{3}{4}, 1\right] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2E_2 + E_3) \text{ if } v \in \left[0, \frac{3}{4}\right] \\ (2v-1)E_2 + (3v-2)E_3 + (4v-3)C \text{ if } v \in \left[\frac{3}{4}, 1\right] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{4v^2}{3} \text{ if } v \in \left[0, \frac{3}{4}\right] \\ 4(v-1)^2 \text{ if } v \in \left[\frac{3}{4}, 1\right] \end{cases} \quad \text{and } P(v) \cdot E_1 = \begin{cases} \frac{4v}{3} \text{ if } v \in \left[0, \frac{3}{4}\right] \\ 4(1-v) \text{ if } v \in \left[\frac{3}{4}, 1\right] \end{cases}$$

We have $S_S(E_1) = \frac{5}{9}$. Thus, $\delta_P(S) \leq \frac{9}{5}$ for $P \in E_1 \backslash E_2$. Moreover, for such points we have

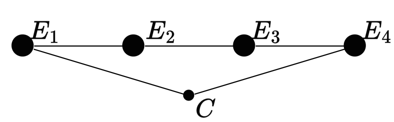
$$h(v) = \begin{cases} \frac{8v^2}{9} \text{ if } v \in \left[0, \frac{3}{4}\right] \\ 4(1-v)(2v-1) \text{ if } v \in \left[\frac{3}{4}, 1\right] \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet}; P) \leq \frac{5}{12} < \frac{7}{12}$. We get $\delta_P(S) = \frac{12}{7}$ for $P \in (E_1 \cup E_3) \backslash E_2$. Thus, $\delta_{\mathcal{P}}(X) = \frac{3}{2}$. □

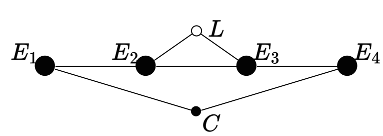
8. $\mathbb{A}_4$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_4$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X) = \frac{4}{3}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E_1, E_2, E_3 and E_4 are the exceptional divisors with the intersection:



We have $-K_S \sim C + E_1 + E_2 + E_3 + E_4$. Let P be a point on S. Consider a linear system $\mathcal{L} = |-2K_S - (E_1 + 2E_2 + 2E_3 + E_4)|$. Using Riemann-Roch for surfaces We get $\dim |\mathcal{L}| = 1$. Thus there is a unique element $L \in |\mathcal{L}|$ such that it contains the intersection point of E_2 and E_3. Moreover we have $L \cdot E_1 = L \cdot E_4 = 0$, $L \cdot E_2 = L \cdot E_3 = 1$ and $L^2 = 0$.



Step 1. Suppose $P = E_2 \cap E_3$. Consider the blowup $\sigma : \widetilde{S} \to S$ of S at P with the exceptional divisor E_P. Suppose $\widetilde{E}_1$, $\widetilde{E}_2$, $\widetilde{E}_3$, $\widetilde{E}_4$, $\widetilde{L}$ and $\widetilde{C}$ are strict transforms of E_1, E_2, E_3, E_4, L and C on $\widetilde{S}$. The Zariski decomposition of the divisor

$$\sigma^*(-K_S) - vE_P \sim \left(\frac{5}{2} - v\right)E_P + \frac{1}{2}\widetilde{L} + \widetilde{E}_1 + \frac{3}{2}\widetilde{E}_2 + \frac{3}{2}\widetilde{E}_3 + \widetilde{E}_4$$

is given by:

$$P(v) = \begin{cases} \sigma^*(-K_S) - vE_P - \frac{v}{5}(\widetilde{E}_1 + 2\widetilde{E}_2 + 2\widetilde{E}_3 + \widetilde{E}_4) \text{ if } v \in \big[0, 2\big] \\ \sigma^*(-K_S) - vE_P - \frac{v}{5}(\widetilde{E}_1 + 2\widetilde{E}_2 + 2\widetilde{E}_3 + \widetilde{E}_4) - (v-2)\widetilde{L} \text{ if } v \in \big[2, \frac{5}{2}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(\widetilde{E}_1 + 2\widetilde{E}_2 + 2\widetilde{E}_3 + \widetilde{E}_4) \text{ if } v \in \big[0, 2\big] \\ \frac{v}{5}(\widetilde{E}_1 + 2\widetilde{E}_2 + 2\widetilde{E}_3 + \widetilde{E}_4) - (v-2)\widetilde{L} \text{ if } v \in \big[2, \frac{5}{2}\big]. \end{cases}$$

Moreover,

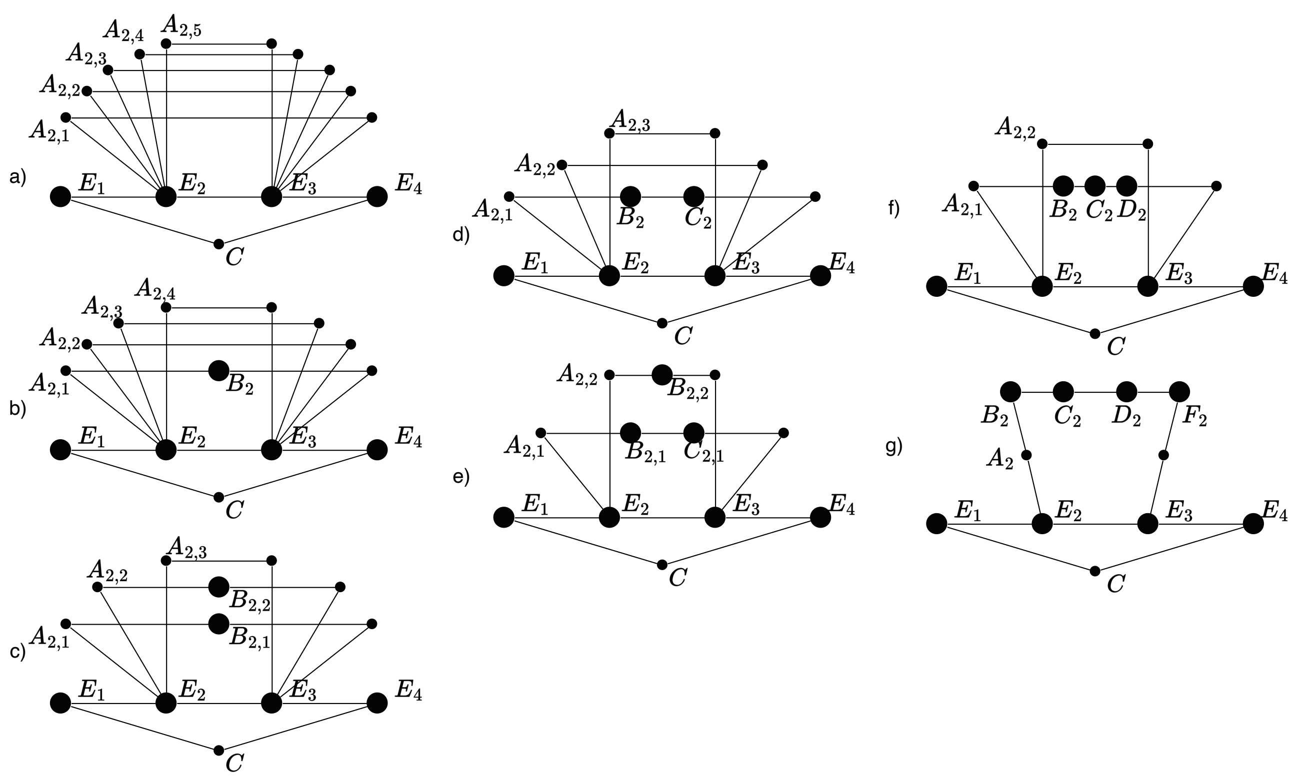
$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{5} \text{ if } v \in \big[0, 2\big] \\ \frac{(5-2v)^2}{5} \text{ if } v \in \big[2, \frac{5}{2}\big] \end{cases} \quad \text{and } P(v) \cdot E_P = \begin{cases} \frac{v}{5} \text{ if } v \in \big[0, 2\big] \\ 2(1 - \frac{2v}{5}) \text{ if } v \in \big[2, \frac{5}{2}\big] \end{cases}$$

We have $S_S(E_P) = \frac{3}{2}$. Thus, $\delta_P(S) \leq \frac{2}{3/2} = \frac{4}{3}$ for $P = E_2 \cap E_3$. Moreover, if $O \in E_P \backslash (\widetilde{E}_2 \cup \widetilde{E}_3)$ if $O \in E_P \backslash \widetilde{L}$ we have:

$$h(v) \leq \begin{cases} \frac{v^2}{50} \text{ if } v \in \big[0, 2\big] \\ \frac{2(5-2v)(3v-5)}{25} \text{ if } v \in \big[2, \frac{5}{2}\big] \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{v^2}{10} \text{ if } v \in \big[0, 2\big] \\ \frac{2(5-2v)}{5} \text{ if } v \in \big[2, \frac{5}{2}\big] \end{cases}$$

Thus, $S(W^{E_P}_{\bullet,\bullet}; O) \leq \frac{1}{6} < \frac{3}{4}$ or $S(W^{E_P}_{\bullet,\bullet}; O) \leq \frac{11}{15} < \frac{3}{4}$. We get $\delta_P(S) = \frac{4}{3}$ for $P = E_2 \cap E_3$.

Step 2. Suppose $P \in E_2 \cup E_3$. Without loss of generality we can assume that $P \in E_2$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:



Then the corresponding Zariski Decomposition of the divisor $-K_S - vE_2$ is:

a).
$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{6}(3E_1 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_2 - \frac{v}{6}(3E_1 + 4E_3 + 2E_4) - (v-1)(A_{2,1} + A_{2,2} + A_{2,3} + A_{2,4} + A_{2,5}) \text{ if } v \in \big[1, \frac{6}{5}\big] \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{6}(3E_1 + 4E_3 + 2E_4) + (v-1)(A_{2,1} + A_{2,2} + A_{2,3} + A_{2,4} + A_{2,5}) \text{ if } v \in \big[1, \frac{6}{5}\big] \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{6}(3E_1 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_2 - \frac{v}{6}(3E_1 + 4E_3 + 2E_4) - (v-1)(2A_{2,1} + B_2 + A_{2,2} + A_{2,3} + A_{2,4}) \text{ if } v \in \big[1, \frac{6}{5}\big] \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{6}(3E_1 + 4E_3 + 2E_4) + (v-1)(2A_{2,1} + B_2 + A_{2,2} + A_{2,3} + A_{2,4}) \text{ if } v \in \big[1, \frac{6}{5}\big] \end{cases}$$

c).
$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{6}(3E_1 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_2 - \frac{v}{6}(3E_1 + 4E_3 + 2E_4) - (v-1)(2A_{2,1} + B_{2,1} + 2A_{2,2} + B_{2,2} + A_{2,3}) \text{ if } v \in \big[1, \frac{6}{5}\big] \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{6}(3E_1 + 4E_3 + 2E_4) + (v-1)(2A_{2,1} + B_{2,1} + 2A_{2,2} + B_{2,2} + A_{2,3}) \text{ if } v \in \big[1, \frac{6}{5}\big] \end{cases}$$

d).
$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{6}(3E_1 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_2 - \frac{v}{6}(3E_1 + 4E_3 + 2E_4) - (v-1)(3A_{2,1} + 2B_2 + C_2 + 2A_{2,2} + A_{2,3}) \text{ if } v \in \big[1, \frac{6}{5}\big] \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{6}(3E_1 + 4E_3 + 2E_4) + (v-1)(3A_{2,1} + 2B_2 + C_2 + A_{2,2} + A_{2,3}) \text{ if } v \in \big[1, \frac{6}{5}\big] \end{cases}$$

e).
$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{6}(3E_1 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_2 - \frac{v}{6}(3E_1 + 4E_3 + 2E_4) - (v-1)(3A_{2,1} + 2B_{2,1} + C_{2,1} + 2A_{2,2} + B_{2,2}) \text{ if } v \in \big[1, \frac{6}{5}\big] \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{6}(3E_1 + 4E_3 + 2E_4) + (v-1)(3A_{2,1} + 2B_{2,1} + C_{2,1} + 2A_{2,2} + B_{2,2}) \text{ if } v \in \big[1, \frac{6}{5}\big] \end{cases}$$

f).
$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{6}(3E_1 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_2 - \frac{v}{6}(3E_1 + 4E_3 + 2E_4) - (v-1)(4A_{2,1} + 3B_2 + 2C_2 + D_2 + A_{2,2}) \text{ if } v \in \big[1, \frac{6}{5}\big] \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{6}(3E_1 + 4E_3 + 2E_4) + (v-1)(4A_{2,1} + 3B_2 + 2C_2 + D_2 + A_{2,2}) \text{ if } v \in \big[1, \frac{6}{5}\big] \end{cases}$$

g).
$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{6}(3E_1 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_2 - \frac{v}{6}(3E_1 + 4E_3 + 2E_4) - (v-1)(5A_2 + 4B_2 + 3C_2 + 2D_2 + F_2) \text{ if } v \in \big[1, \frac{6}{5}\big] \end{cases}$$
$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{6}(3E_1 + 4E_3 + 2E_4) + (v-1)(5A_2 + 4B_2 + 3C_2 + 2D_2 + F_2) \text{ if } v \in \big[1, \frac{6}{5}\big] \end{cases}$$

The Zariski Decomposition in part a). follows from
$$-K_S - vE_2 \sim_{\mathbb{R}} \Big(\frac{6}{5} - v\Big)E_2 + \frac{1}{5}\Big(3E_1 + 4E_3 + 2E_4 + A_{2,1} + A_{2,2} + A_{2,3} + A_{2,4} + A_{2,5}\Big)$$
A similar statement holds in other parts. Moreover,
$$(P(v))^2 = \begin{cases} 1 - \frac{5v^2}{6} \text{ if } v \in \big[0, 1\big] \\ \frac{(6-5v)^2}{6} \text{ if } v \in \big[1, \frac{6}{5}\big] \end{cases} \quad \text{and } P(v) \cdot E_2 = \begin{cases} \frac{5v}{6} \text{ if } v \in \big[0, 1\big] \\ 3(1-v) \text{ if } v \in \big[1, \frac{6}{5}\big] \end{cases}$$

We have $S_S(E_2)=\frac{11}{15}$. Thus, $\delta_P(S)\le\frac{15}{11}$ for $P\in E_2\backslash E_3$. Moreover, if $P\in E_2\cap E_1$ or if $P\in E_2\backslash E_1$ for such points we have

$$h(v)\le\begin{cases}\frac{55v^2}{72} \text{ if } v\in\big[0,1\big]\\ \frac{5(5v-6)(19v-30)}{72} \text{ if } v\in\big[1,\frac{6}{5}\big]\end{cases} \quad \text{or } h(v)\le\begin{cases}\frac{25v^2}{72} \text{ if } v\in\big[0,1\big]\\ \frac{25(5v-6)(6-7v)}{72} \text{ if } v\in\big[1,\frac{6}{5}\big]\end{cases}$$

Thus, $S(W^{E_2}_{\bullet,\bullet};P)\le\frac{29}{45}<\frac{11}{15}$ or $S(W^{E_2}_{\bullet,\bullet};P)\frac{1}{3}<\frac{11}{15}$. We get $\delta_P(S)=\frac{15}{11}$ for $P\in(E_1\cup E_2)\backslash(E_1\cap E_2)$.

Step 3. Suppose $P\in E_1\cup E_4$. Without loss of generality we can assume that $P\in E_1$ since the proof is similar in other cases. The Zariski decomposition of the divisor $-K_S-vE_1\sim C+(1-v)E_1+E_2+E_3+E_4$ is given by:

$$P(v)=\begin{cases}-K_S-vE_1-\frac{v}{4}(3E_2+2E_3+E_4) \text{ if } v\in\big[0,\frac{4}{5}\big]\\ -K_S-vE_1-(2v-1)E_2-(3v-2)E_3-(4v-3)E_4-(5v-4)C \text{ if } v\in\big[\frac{4}{5},1\big]\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{4}(3E_2+2E_3+E_4) \text{ if } v\in\big[0,\frac{4}{5}\big]\\ (2v-1)E_2+(3v-2)E_3+(4v-3)E_4+(5v-4)C \text{ if } v\in\big[\frac{4}{5},1\big]\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{5v^2}{4} \text{ if } v\in\big[0,\frac{4}{5}\big]\\ 5(v-1)^2 \text{ if } v\in\big[\frac{4}{5},1\big]\end{cases} \quad \text{and } P(v)\cdot E_1=\begin{cases}\frac{5v}{4} \text{ if } v\in\big[0,\frac{4}{5}\big]\\ 5(1-v) \text{ if } v\in\big[\frac{4}{5},1\big]\end{cases}$$

We have $S_S(E_1)=\frac{3}{5}$ Thus, $\delta_P(S)\le\frac{5}{3}$ for $P\in E_1\backslash E_2$. Moreover, for such points we have

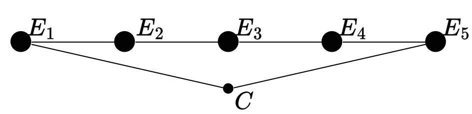
$$h(v)=\begin{cases}\frac{25v^2}{32} \text{ if } v\in\big[0,\frac{4}{5}\big]\\ \frac{5(1-v)(5v-3)}{2} \text{ if } v\in\big[\frac{4}{5},1\big]\end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet};P)\le\frac{2}{5}<\frac{3}{5}$. We get $\delta_P(S)=\frac{5}{3}$ for $P\in(E_1\cup E_4)\backslash(E_2\cup E_3)$. Thus, $\delta_{\mathcal{P}}(X)=\frac{4}{3}$. □

9. $\mathbb{A}_5$ singularity on Du Val Del Pezzo surfaces of degree 1

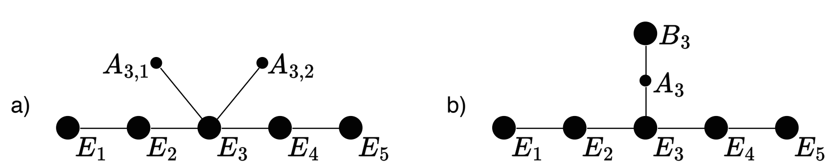
Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_5$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X)=\frac{6}{5}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E_1, E_2, E_3, E_4 and E_5 are the exceptional divisors with the intersection:



We have $-K_S\sim C+E_1+E_2+E_3+E_4+E_5$. Let P be a point on S.

Step 1. Suppose $P\in E_3$. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:



Then the corresponding Zariski Decomposition of the divisor $-K_S - vE_2$ is:

a). $$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_3 - \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) - (v-1)(A_{3,1} + A_{3,2}) \text{ if } v \in \big[1, \frac{3}{2}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) + (v-1)(A_{3,1} + A_{3,2}) \text{ if } v \in \big[1, \frac{3}{2}\big] \end{cases}$$

b). $$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_3 - \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) - (v-1)(2A_3 + B_3) \text{ if } v \in \big[1, \frac{3}{2}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{3}(E_1 + 2E_2 + 2E_4 + E_5) + (v-1)(2A_3 + B_3) \text{ if } v \in \big[1, \frac{3}{2}\big] \end{cases}$$

The Zariski Decomposition in part a). follows from

$$-K_S - vE_3 \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)E_3 + \frac{1}{2}\Big(E_1 + 2E_2 + 2E_4 + E_5 + A_{3,1} + A_{3,2}\Big)$$

A similar statement holds in other parts. Moreover,

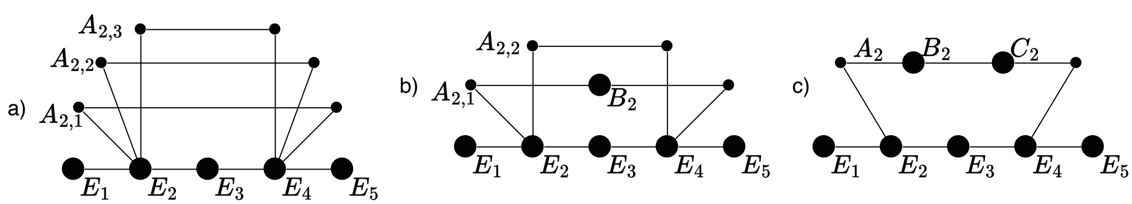
$$(P(v))^2 = \begin{cases} 1 - \frac{2v^2}{3} \text{ if } v \in \big[0, 1\big] \\ \frac{(3-2v)^2}{3} \text{ if } v \in \big[1, \frac{3}{2}\big] \end{cases} \quad \text{and } P(v) \cdot E_3 = \begin{cases} \frac{2v}{3} \text{ if } v \in \big[0, 1\big] \\ 2(1 - \frac{2v}{3}) \text{ if } v \in \big[1, \frac{3}{2}\big] \end{cases}$$

We have $S_S(E_3) = \frac{5}{6}$. Thus, $\delta_P(S) \leq \frac{6}{5}$ for $P \in E_3$. Moreover, if $P \in E_3 \cap (E_2 \cup E_4)$ or if $P \in E_3 \backslash (E_2 \cup E_4)$ we have

$$h(v) \leq \begin{cases} \frac{2v^2}{3} \text{ if } v \in \big[0, 1\big] \\ \frac{2(3-2v)}{3} \text{ if } v \in \big[1, \frac{3}{2}\big] \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{2v^2}{9} \text{ if } v \in \big[0, 1\big] \\ \frac{2(3-2v)(4v-3)}{9} \text{ if } v \in \big[1, \frac{3}{2}\big] \end{cases}$$

Thus, $S(W^{E_3}_{\bullet,\bullet}; P) \leq \frac{7}{9} < \frac{5}{6}$ or $S(W^{E_3}_{\bullet,\bullet}; P) \leq \frac{1}{3} < \frac{5}{6}$. We get $\delta_P(S) = \frac{6}{5}$ for $P \in E_3$.

Step 2. Suppose $P \in E_2 \cup E_4$. Without loss of generality we can assume that $P \in E_2$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:



Then the corresponding Zariski Decomposition of the divisor $-K_S - vE_2$ is:

a). $$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{4}(2E_1 + 3E_3 + 2E_4 + E_5) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_2 - \frac{v}{4}(2E_1 + 3E_3 + 2E_4 + E_5) - (v-1)(A_{2,1} + A_{2,2} + A_{2,3}) \text{ if } v \in \big[1, \frac{4}{3}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(2E_1 + 3E_3 + 2E_4 + E_5) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{4}(2E_1 + 3E_3 + 2E_4 + E_5) + (v-1)(A_{2,1} + A_{2,2} + A_{2,3}) \text{ if } v \in \big[1, \frac{4}{3}\big] \end{cases}$$

b). $$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{4}(2E_1 + 3E_3 + 2E_4 + E_5) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_2 - \frac{v}{4}(2E_1 + 3E_3 + 2E_4 + E_5) - (v-1)(2A_{2,1} + B_{2,1} + A_{2,2}) \text{ if } v \in \big[1, \frac{4}{3}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(2E_1 + 3E_3 + 2E_4 + E_5) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{4}(2E_1 + 3E_3 + 2E_4 + E_5) + (v-1)(2A_{2,1} + B_{2,1} + A_{2,2}) \text{ if } v \in \big[1, \frac{4}{3}\big] \end{cases}$$

c). $$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{4}(2E_1 + 3E_3 + 2E_4 + E_5) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_2 - \frac{v}{4}(2E_1 + 3E_3 + 2E_4 + E_5) - (v-1)(3A_2 + B_2 + C_2) \text{ if } v \in \big[1, \frac{4}{3}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(2E_1 + 3E_3 + 2E_4 + E_5) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{4}(2E_1 + 3E_3 + 2E_4 + E_5) + (v-1)(3A_2 + B_2 + C_2) \text{ if } v \in \big[1, \frac{4}{3}\big] \end{cases}$$

The Zariski Decomposition in part a). follows from

$$-K_S - vE_2 \sim_{\mathbb{R}} \Big(\frac{4}{3} - v\Big)E_2 + \frac{1}{3}\Big(2E_1 + 3E_3 + 2E_4 + E_5 + A_{2,1} + A_{2,2} + A_{2,3}\Big)$$

A similar statement holds in other parts. Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{3v^2}{4} \text{ if } v \in \big[0, 1\big] \\ \frac{(4-3v)^2}{4} \text{ if } v \in \big[1, \frac{4}{3}\big] \end{cases} \quad \text{and } P(v) \cdot E_2 = \begin{cases} \frac{3v}{4} \text{ if } v \in \big[0, 1\big] \\ 3(1 - \frac{3v}{4}) \text{ if } v \in \big[1, \frac{4}{3}\big] \end{cases}$$

We have $S_S(E_2) = \frac{7}{9}$. Thus, $\delta_P(S) \le \frac{9}{7}$ for $P \in E_2$. Moreover, if $P \in E_2 \cap E_1$ or if $P \in E_2 \backslash (E_1 \cup E_3)$ we have

$$h(v) \le \begin{cases} \frac{21v^2}{32} \text{ if } v \in \big[0, 1\big] \\ \frac{3(3v-4)(5v-12)}{32} \text{ if } v \in \big[1, \frac{4}{3}\big] \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{9v^2}{32} \text{ if } v \in \big[0, 1\big] \\ \frac{9(3v-4)(4-5v)}{32} \text{ if } v \in \big[1, \frac{4}{3}\big] \end{cases}$$

Thus, $S(W^{E_2}_{\bullet,\bullet}; P) \le \frac{23}{36} < \frac{7}{9}$ or $S(W^{E_2}_{\bullet,\bullet}; P) \le \frac{1}{3} < \frac{7}{9}$. We get $\delta_P(S) = \frac{9}{7}$ for $P \in (E_2 \cup E_4) \backslash E_3$.

Step 3. Suppose $P \in E_1 \cup E_5$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. The Zariski decomposition of the divisor $-K_S - vE_1 \sim C + (1 - v)E_1 + E_2 + E_3 + E_4 + E_5$ is given by:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{5}(4E_2 + 3E_3 + 2E_4 + E_5) \text{ if } v \in \big[0, \frac{5}{6}\big] \\ -K_S - vE_1 - (2v-1)E_2 - (3v-2)E_3 - (4v-3)E_4 - (5v-4)E_5 - (6v-5)C \text{ if } v \in \big[\frac{5}{6}, 1\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(4E_2 + 3E_3 + 2E_4 + E_5) \text{ if } v \in \big[0, \frac{5}{6}\big] \\ (2v-1)E_2 + (3v-2)E_3 + (4v-3)E_4 + (5v-4)E_5 + (6v-5)C \text{ if } v \in \big[\frac{5}{6}, 1\big] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{6v^2}{5} \text{ if } v \in \big[0, \frac{5}{6}\big] \\ 6(v-1)^2 \text{ if } v \in \big[\frac{5}{6}, 1\big] \end{cases} \quad \text{and } P(v) \cdot E_1 = \begin{cases} \frac{6v}{5} \text{ if } v \in \big[0, \frac{5}{6}\big] \\ 6(1-v) \text{ if } v \in \big[\frac{5}{6}, 1\big] \end{cases}$$

We have $S_S(E_1)=\frac{11}{18}$. Thus, $\delta_P(S)\leq\frac{18}{11}$ for $P\in E_1\backslash E_2$. Moreover, for such points we have

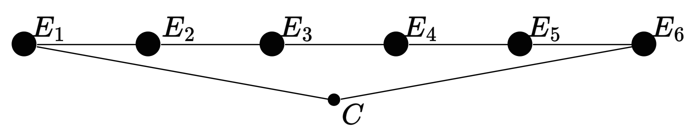
$$h(v)=\begin{cases}\frac{18v^2}{25} \text{ if } v\in\big[0,\frac{5}{6}\big]\\ 6(1-v)(3v-2) \text{ if } v\in\big[\frac{5}{6},1\big]\end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet};P)\leq\frac{7}{18}<\frac{11}{18}$. We get $\delta_P(S)=\frac{18}{11}$ for $P\in(E_1\cup E_5)\backslash(E_2\cup E_4)$. Thus, $\delta_{\mathcal{P}}(X)=\frac{6}{5}$. □

10. $\mathbb{A}_6$ singularity on Du Val Del Pezzo surfaces of degree 1

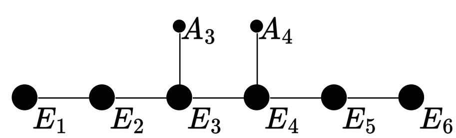
Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_6$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X)=\frac{9}{8}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E_1, E_2, E_3, E_4, E_6 and E_7 are the exceptional divisors with the intersection:



We have $-K_S\sim C+E_1+E_2+E_3+E_4+E_5+E_6$. Let P be a point on S.

Step 1. Suppose $P\in E_3\cup E_4$. Without loss of generality we can assume that $P\in E_3$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form the following dual graph:



Then the corresponding Zariski Decomposition of the divisor $-K_S-vE_3$ is:

$$P(v)=\begin{cases}-K_S-vE_3-\frac{v}{12}(4E_1+8E_2+9E_4+6E_5+3E_6) \text{ if } v\in\big[0,1\big]\\ -K_S-vE_3-\frac{v}{12}(4E_1+8E_2+9E_4+6E_5+3E_6)-(v-1)A_3 \text{ if } v\in\big[1,\frac{4}{3}\big]\\ -K_S-vE_3-\frac{v}{3}(E_1+2E_2)-(v-1)(3E_4+2E_5+E_6+A_3)-(3v-4)A_4 \text{ if } v\in\big[\frac{4}{3},\frac{3}{2}\big]\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{12}(4E_1+8E_2+9E_4+6E_5+3E_6) \text{ if } v\in\big[0,1\big]\\ \frac{v}{12}(4E_1+8E_2+9E_4+6E_5+3E_6)+(v-1)A_3 \text{ if } v\in\big[1,\frac{4}{3}\big]\\ \frac{v}{3}(E_1+2E_2)+(v-1)(3E_4+2E_5+E_6+A_3)+(3v-4)A_4 \text{ if } v\in\big[\frac{4}{3},\frac{3}{2}\big]\end{cases}$$

The Zariski Decomposition in part a). follows from

$$-K_S-vE_3\sim_{\mathbb{R}}\Big(\frac{3}{2}-v\Big)E_3+\frac{1}{2}\Big(E_1+2E_2+3E_4+2E_5+E_6+A_3+A_4\Big)$$

Moreover,

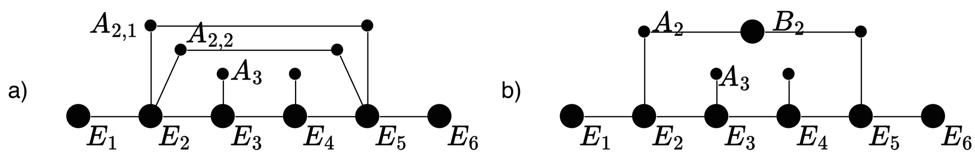
$$(P(v))^2=\begin{cases}1-\frac{7v^2}{12} \text{ if } v\in\big[0,1\big]\\ 2-2v+\frac{5v^2}{12} \text{ if } v\in\big[1,\frac{4}{3}\big]\\ \frac{2(3-2v)^2}{3} \text{ if } v\in\big[\frac{4}{3},\frac{3}{2}\big]\end{cases} \quad\text{and } P(v)\cdot E_3=\begin{cases}\frac{7v}{12} \text{ if } v\in\big[0,1\big]\\ 1-\frac{5v}{12} \text{ if } v\in\big[1,\frac{4}{3}\big]\\ 4(1-\frac{2v}{3}) \text{ if } v\in\big[\frac{4}{3},\frac{3}{2}\big]\end{cases}$$

We have $S_S(E_3) = \frac{8}{9}$. Thus, $\delta_P(S) \leq \frac{9}{8}$ for $P \in E_3$. Moreover, if $P \in E_3 \cap A_3$ or if $P \in E_3 \cap E_2$ or if $P \in E_3 \backslash (E_2 \cup A_3)$ we have

$$h(v) \leq \begin{cases} \frac{49v^2}{288} \text{ if } v \in \big[0,1\big] \\ \frac{(12-5v)(19v-12)}{288} \text{ if } v \in \big[1,\frac{4}{3}\big] \\ \frac{4(2v-3)(v-3)}{9} \text{ if } v \in \big[\frac{4}{3},\frac{3}{2}\big] \end{cases} \text{ or } h(v) \leq \begin{cases} \frac{161v^2}{288} \text{ if } v \in \big[0,1\big] \\ \frac{(12-5v)(11v+12)}{288} \text{ if } v \in \big[1,\frac{4}{3}\big] \\ \frac{8(2v-3)(v-3)}{9} \text{ if } v \in \big[\frac{4}{3},\frac{3}{2}\big] \end{cases} \text{ or } h(v) \leq \begin{cases} \frac{175v^2}{288} \text{ if } v \in \big[0,1\big] \\ \frac{(12-5v)(13v+12)}{288} \text{ if } v \in \big[1,\frac{4}{3}\big] \\ \frac{4(2v-3)(5v-3)}{9} \text{ if } v \in \big[\frac{4}{3},\frac{3}{2}\big] \end{cases}$$

Thus, $S(W^{E_3}_{\bullet,\bullet}; P) \leq \frac{8}{27} < \frac{8}{9}$ or $S(W^{E_3}_{\bullet,\bullet}; P) \leq \frac{29}{36} < \frac{8}{9}$ or $S(W^{E_3}_{\bullet,\bullet}; P) \leq \frac{8}{9}$. We get $\delta_P(S) = \frac{9}{8}$ for $P \in E_3 \cup E_4$.

Step 2. Suppose $P \in E_2 \cup E_5$. Without loss of generality we can assume that $P \in E_2$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:



Then the corresponding Zariski Decomposition of the divisor $-K_S - vE_2$ is:

a).
$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{10}(5E_1 + 8E_3 + 6E_4 + 4E_5 + 2E_6) \text{ if } v \in \big[0,1\big] \\ -K_S - vE_2 - \frac{v}{10}(5E_1 + 8E_3 + 6E_4 + 4E_5 + 2E_6) - (v-1)(A_{2,1} + A_{2,2}) \text{ if } v \in \big[1,\frac{5}{4}\big] \\ -K_S - vE_2 - \frac{v}{2}E_1 - (v-1)(4E_3 + 3E_4 + 2E_5 + E_6 + A_{2,1} + A_{2,2}) - (4v-5)A_3 \text{ if } v \in \big[\frac{5}{4},\frac{4}{3}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{10}(5E_1 + 8E_3 + 6E_4 + 4E_5 + 2E_6) \text{ if } v \in \big[0,1\big] \\ \frac{v}{10}(5E_1 + 8E_3 + 6E_4 + 4E_5 + 2E_6) + (v-1)(A_{2,1} + A_{2,2}) \text{ if } v \in \big[1,\frac{5}{4}\big] \\ \frac{v}{2}E_1 + (v-1)(4E_3 + 3E_4 + 2E_5 + E_6 + A_{2,1} + A_{2,2}) + (4v-5)A_3 \text{ if } v \in \big[\frac{5}{4},\frac{4}{3}\big] \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{10}(5E_1 + 8E_3 + 6E_4 + 4E_5 + 2E_6) \text{ if } v \in \big[0,1\big] \\ -K_S - vE_2 - \frac{v}{10}(5E_1 + 8E_3 + 6E_4 + 4E_5 + 2E_6) - (v-1)(2A_2 + B_2) \text{ if } v \in \big[1,\frac{5}{4}\big] \\ -K_S - vE_2 - \frac{v}{2}E_1 - (v-1)(4E_3 + 3E_4 + 2E_5 + E_6 + 2A_2 + B_2) - (4v-5)A_3 \text{ if } v \in \big[\frac{5}{4},\frac{4}{3}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{10}(5E_1 + 8E_3 + 6E_4 + 4E_5 + 2E_6) \text{ if } v \in \big[0,1\big] \\ \frac{v}{10}(5E_1 + 8E_3 + 6E_4 + 4E_5 + 2E_6) + (v-1)(2A_2 + B_2) \text{ if } v \in \big[1,\frac{5}{4}\big] \\ \frac{v}{2}E_1 + (v-1)(4E_3 + 3E_4 + 2E_5 + E_6 + 2A_2 + B_2) + (4v-5)A_3 \text{ if } v \in \big[\frac{5}{4},\frac{4}{3}\big] \end{cases}$$

The Zariski Decomposition in part a). follows from

$$-K_S - vE_2 \sim_{\mathbb{R}} \Big(\frac{4}{3} - v\Big)E_2 + \frac{1}{3}\Big(2E_1 + 4E_3 + 3E_4 + 2E_5 + E_6 + A_{2,1} + A_{2,2} + A_3\Big)$$

A similar statement holds in other parts. Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{7v^2}{10} \text{ if } v \in \big[0,1\big] \\ 3 - 4v + \frac{13v^2}{10} \text{ if } v \in \big[1,\frac{5}{4}\big] \\ \frac{(4-3v)^2}{2} \text{ if } v \in \big[\frac{5}{4},\frac{4}{3}\big] \end{cases} \text{ and } P(v) \cdot E_2 = \begin{cases} \frac{7v}{10} \text{ if } v \in \big[0,1\big] \\ 2 - \frac{13v}{10} \text{ if } v \in \big[1,\frac{5}{4}\big] \\ 3(2 - \frac{3v}{2}) \text{ if } v \in \big[\frac{5}{4},\frac{4}{3}\big] \end{cases}$$

We have $S_S(E_2)=\frac{29}{36}$. Thus, $\delta_P(S)\le\frac{36}{29}$ for $P\in E_2$. Moreover, if $P\in E_2\cap E_1$ or if $P\in E_2\backslash(E_1\cup E_3)$ we have

$$h(v)\le\begin{cases}\frac{119v^2}{200}\text{ if }v\in\big[0,1\big]\\\frac{(13v-20)(3v-20)}{200}\text{ if }v\in\big[1,\frac{5}{4}\big]\\\frac{3(3v-4)(7v-12)}{8}\text{ if }v\in\big[\frac{5}{4},\frac{4}{3}\big]\end{cases}\quad\text{or } h(v)\le\begin{cases}\frac{49v^2}{200}\text{ if }v\in\big[0,1\big]\\\frac{(13v-20)(27v-20)}{200}\text{ if }v\in\big[1,\frac{5}{4}\big]\\\frac{3(3v-4)(v-4)}{8}\text{ if }v\in\big[\frac{5}{4},\frac{4}{3}\big]\end{cases}$$

Thus $S(W^{E_2}_{\bullet,\bullet};P)\le\frac{29}{45}<\frac{29}{36}$ or $S(W^{E_2}_{\bullet,\bullet};P)\le\frac{23}{72}<\frac{29}{36}$. We get $\delta_P(S)=\frac{36}{29}$ for $P\in(E_2\cup E_5)\backslash(E_3\cup E_4)$.
Step 3. Suppose $P\in E_1\cup E_6$. Without loss of generality we can assume that $P\in E_1$ since the proof is similar in other cases. The Zariski decomposition of the divisor $-K_S-vE_1\sim C+(1-v)E_1+E_2+E_3+E_4+E_5+E_6$ is given by:

$$P(v)=\begin{cases}-K_S-vE_1-\frac{v}{6}(5E_2+4E_3+3E_4+2E_5+E_6)\text{ if }v\in\big[0,\frac{6}{7}\big]\\-K_S-vE_1-(2v-1)E_2-(3v-2)E_3-(4v-3)E_4-(5v-4)E_5-(6v-5)E_6-(7v-6)C\text{ if }v\in\big[\frac{6}{7},1\big]\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{6}(5E_2+4E_3+3E_4+2E_5+E_6)\text{ if }v\in\big[0,\frac{6}{7}\big]\\(2v-1)E_2+(3v-2)E_3+(4v-3)E_4+(5v-4)E_5+(6v-5)E_6+(7v-6)C\text{ if }v\in\big[\frac{6}{7},1\big]\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{7v^2}{6}\text{ if }v\in\big[0,\frac{6}{7}\big]\\7(v-1)^2\text{ if }v\in\big[\frac{6}{7},1\big]\end{cases}\quad\text{and } P(v)\cdot E_1=\begin{cases}\frac{7v}{6}\text{ if }v\in\big[0,\frac{6}{7}\big]\\7(1-v)\text{ if }v\in\big[\frac{6}{7},1\big]\end{cases}$$

We have $S_S(E_1)=\frac{13}{21}$. Thus, $\delta_P(S)\le\frac{21}{13}$ for $P\in E_1\backslash E_2$. Moreover, for such points we have

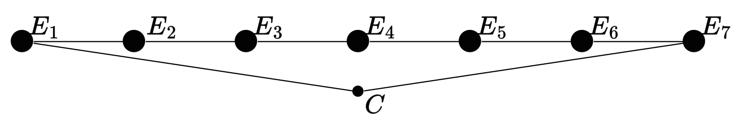
$$h(v)=\begin{cases}\frac{49v^2}{72}\text{ if }v\in\big[0,\frac{6}{7}\big]\\\frac{7(1-v)(7v-5)}{2}\text{ if }v\in\big[\frac{6}{7},1\big]\end{cases}$$

Thus $S(W^{E_1}_{\bullet,\bullet};P)\le\frac{8}{21}<\frac{13}{21}$. We get $\delta_P(S)=\frac{21}{13}$ for $P\in(E_1\cup E_6)\backslash(E_2\cup E_5)$. Thus, $\delta_{\mathcal{P}}(X)=\frac{9}{8}$. □

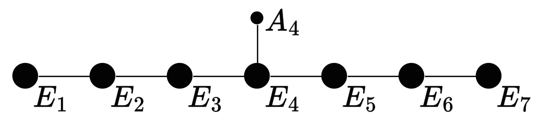
11. $\mathbb{A}_7$ singularity with reducible ramification divisor on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_7$ singularity at point $\mathcal{P}$. X can be realized as the double cover $X\xrightarrow{2:1}\mathbb{P}(1,1,2)$, which is ramified along a sextic curve $R\in\mathbb{P}(1,1,2)$. Suppose R is reducible. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X)=1$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E_1, E_2, E_3, E_4, E_5, E_6 and E_7 are the exceptional divisors with the intersection:



We have $-K_S\sim C+E_1+E_2+E_3+E_4+E_5+E_6+E_7$. Let P be a point on S. If the ramification divisor R is reducible, then this implies the existence of a (-1)-curve A_4 which intersects the fundamental cycle only at E_4 and this intersection is transversal.
Step 1. Suppose $P\in E_4$. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form the following dual graph:



Then the corresponding Zariski Decomposition of the divisor $-K_S - vE_4$ is:

$$P(v) = \begin{cases} -K_S - vE_4 - \frac{v}{4}(E_1 + 2E_2 + 3E_3 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_4 - \frac{v}{4}(E_1 + 2E_2 + 3E_3 + 3E_5 + 2E_6 + E_7) - (v-1)A_4 \text{ if } v \in \big[1, 2\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(E_1 + 2E_2 + 3E_3 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{4}(E_1 + 2E_2 + 3E_3 + 3E_5 + 2E_6 + E_7) + (v-1)A_4 \text{ if } v \in \big[1, 2\big] \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_4 \sim_{\mathbb{R}} (2-v)E_4 + \frac{1}{2}\Big(E_1 + 2E_2 + 3E_3 + 3E_5 + 2E_6 + E_7 + 2A_4\Big)$$

Moreover,

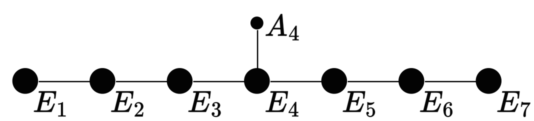
$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{2} \text{ if } v \in \big[0, 1\big] \\ \frac{(2-v)^2}{2} \text{ if } v \in \big[1, 2\big] \end{cases} \quad \text{and } P(v) \cdot E_4 = \begin{cases} \frac{v}{2} \text{ if } v \in \big[0, 1\big] \\ 1 - \frac{v}{2}) \text{ if } v \in \big[1, 2\big] \end{cases}$$

We have $S_S(E_4) = 1$. Thus, $\delta_P(S) \leq 1$ for $P \in E_4$. Moreover, if $P \in E_4 \cap (E_3 \cup E_5)$ or if $P \in E_4 \backslash (E_3 \cup E_5)$ we have

$$h(v) \leq \begin{cases} \frac{v^2}{2} \text{ if } v \in \big[0, 1\big] \\ \frac{(2-v)(v+1)}{4} \text{ if } v \in \big[1, 2\big] \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{v^2}{8} \text{ if } v \in \big[0, 1\big] \\ \frac{(2-v)(3v-2)}{8} \text{ if } v \in \big[1, 2\big] \end{cases}$$

Thus $S(W^{E_4}_{\bullet,\bullet}; P) \leq \frac{11}{12} < 1$ or $S(W^{E_4}_{\bullet,\bullet}; P) \leq \frac{1}{3} < 1$. We get $\delta_P(S) = 1$ for $P \in E_4$.

Step 2. Suppose $P \in E_3 \cup E_5$. Without loss of generality we can assume that $P \in E_3$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form the following dual graph:



Then the corresponding Zariski Decomposition of the divisor $-K_S - vE_3$ is:

$$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{15}(5E_1 + 10E_2 + 12E_4 + 9E_5 + 6E_6 + 3E_7) \text{ if } v \in \big[0, \frac{5}{4}\big] \\ -K_S - vE_3 - \frac{v}{3}(E_1 + 2E_2) - (v-1)(4E_4 + 3E_5 + 2E_6 + E_7) - (4v-5)A_4 \text{ if } v \in \big[\frac{5}{4}, \frac{3}{2}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{15}(5E_1 + 10E_2 + 12E_4 + 9E_5 + 6E_6 + 3E_7) \text{ if } v \in \big[0, \frac{5}{4}\big] \\ \frac{v}{3}(E_1 + 2E_2) + (v-1)(4E_4 + 3E_5 + 2E_6 + E_7) + (4v-5)A_4 \text{ if } v \in \big[\frac{5}{4}, \frac{3}{2}\big] \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_3 \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)E_3 + \frac{1}{2}\Big(E_1 + 2E_2 + 4E_4 + 3E_5 + 2E_6 + E_7 + 2A_4\Big)$$

Moreover,

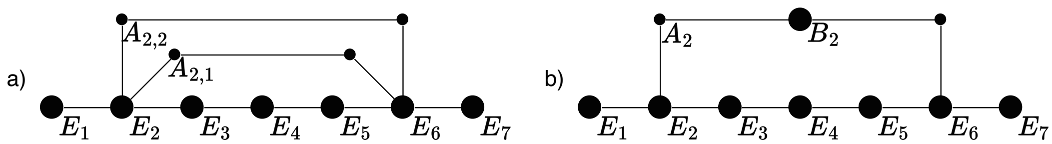
$$(P(v))^2 = \begin{cases} 1-\frac{8v^2}{15} \text{ if } v\in\big[0,\frac{5}{4}\big] \\ \frac{2(3-2v)^2}{3} \text{ if } v\in\big[\frac{5}{4},\frac{3}{2}\big] \end{cases} \quad \text{and } P(v)\cdot E_3 = \begin{cases} \frac{8v}{15} \text{ if } v\in\big[0,\frac{5}{4}\big] \\ 4(1-\frac{2v}{3}) \text{ if } v\in\big[\frac{5}{4},\frac{3}{2}\big] \end{cases}$$

We have $S_S(E_3)=\frac{11}{12}$. Thus, $\delta_P(S)\le\frac{12}{11}$ for $P\in E_3$. Moreover, if $P\in E_3\backslash E_4$ we have

$$h(v)\le \begin{cases} \frac{112v^2}{225} \text{ if } v\in\big[0,\frac{5}{4}\big] \\ \frac{8(2v-3)(v-3)}{9} \text{ if } v\in\big[\frac{5}{4},\frac{3}{2}\big] \end{cases}$$

Thus, $S(W^{E_3}_{\bullet,\bullet};P)\le\frac{5}{6}<\frac{11}{12}$. We get $\delta_P(S)=\frac{12}{11}$ for $P\in(E_3\cup E_5)\backslash E_4$.

Step 3. Suppose $P\in E_2\cup E_6$. Without loss of generality we can assume that $P\in E_2$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:



Then the corresponding Zariski Decomposition of the divisor $-K_S-vE_2$ is:

a). $$P(v)=\begin{cases} -K_S-vE_2-\frac{v}{6}(3E_1+5E_3+4E_4+3E_5+2E_6+E_7) \text{ if } v\in\big[0,1\big] \\ -K_S-vE_2-\frac{v}{6}(3E_1+5E_3+4E_4+3E_5+2E_6+E_7)-(v-1)(A_{2,1}+A_{2,2}) \text{ if } v\in\big[1,\frac{3}{2}\big] \end{cases}$$

$$N(v)=\begin{cases} \frac{v}{6}(3E_1+5E_3+4E_4+3E_5+2E_6+E_7) \text{ if } v\in\big[0,1\big] \\ \frac{v}{6}(3E_1+5E_3+4E_4+3E_5+2E_6+E_7)+(v-1)(A_{2,1}+A_{2,2}) \text{ if } v\in\big[1,\frac{3}{2}\big] \end{cases}$$

b). $$P(v)=\begin{cases} -K_S-vE_2-\frac{v}{6}(3E_1+5E_3+4E_4+3E_5+2E_6+E_7) \text{ if } v\in\big[0,1\big] \\ -K_S-vE_2-\frac{v}{6}(3E_1+5E_3+4E_4+3E_5+2E_6+E_7)-(v-1)(2A_2+B_2) \text{ if } v\in\big[1,\frac{3}{2}\big] \end{cases}$$

$$N(v)=\begin{cases} \frac{v}{6}(3E_1+5E_3+4E_4+3E_5+2E_6+E_7) \text{ if } v\in\big[0,1\big] \\ \frac{v}{6}(3E_1+5E_3+4E_4+3E_5+2E_6+E_7)+(v-1)(2A_2+B_2) \text{ if } v\in\big[1,\frac{3}{2}\big] \end{cases}$$

The Zariski Decomposition follows from

$$-K_S-vE_2\sim_{\mathbb{R}}\Big(\frac{3}{2}-v\Big)E_2+\frac{1}{4}\Big(3E_1+5E_3+4E_4+3E_5+2E_6+E_7+2A_{2,1}+2A_{2,2}\Big)$$

Moreover,

$$(P(v))^2 = \begin{cases} 1-\frac{2v^2}{3} \text{ if } v\in\big[0,1\big] \\ \frac{(3-2v)^2}{3} \text{ if } v\in\big[1,\frac{3}{2}\big] \end{cases} \quad \text{and } P(v)\cdot E_2 = \begin{cases} \frac{2v}{3} \text{ if } v\in\big[0,1\big] \\ 2(1-\frac{2v}{3}) \text{ if } v\in\big[1,\frac{3}{2}\big] \end{cases}$$

We have $S_S(E_2)=\frac{5}{6}$. Thus, $\delta_P(S)\le\frac{6}{5}$ for $P\in E_2$. Moreover, if $P\in E_2\cap E_1$ or if $P\in E_2\backslash(E_1\cup E_3)$ we have

$$h(v)\le \begin{cases} \frac{5v^2}{9} \text{ if } v\in\big[0,1\big] \\ \frac{(2v-3)(v-6)}{9} \text{ if } v\in\big[1,\frac{3}{2}\big] \end{cases} \quad \text{or } h(v)\le \begin{cases} \frac{2v^2}{9} \text{ if } v\in\big[0,1\big] \\ \frac{2(3-2v)(4v-3)}{9} \text{ if } v\in\big[1,\frac{3}{2}\big] \end{cases}$$

Thus $S(W^{E_2}_{\bullet,\bullet};P)\le\frac{23}{36}<\frac{5}{6}$ or $S(W^{E_2}_{\bullet,\bullet};P)\le\frac{1}{3}<\frac{5}{6}$. We get $\delta_P(S)=\frac{6}{5}$ for $P\in(E_2\cup E_6)\backslash(E_1\cup E_7)$.

Step 4. Suppose $P\in E_1\cup E_7$. Without loss of generality we can assume that $P\in E_1$ since the proof is similar in other cases. The Zariski decomposition of the divisor $-K_S - vE_1$ is given by:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{7}(6E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \left[0, \frac{7}{8}\right] \\ -K_S - vE_1 - (2v-1)E_2 - (3v-2)E_3 - (4v-3)E_4 - (5v-4)E_5 - (6v-5)E_6 - (7v-6)E_7 - (8v-7)C \text{ if } v \in \left[\frac{7}{8}, 1\right] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{7}(6E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \left[0, \frac{7}{8}\right] \\ (2v-1)E_2 + (3v-2)E_3 - (4v-3)E_4 + (5v-4)E_5 + (6v-5)E_6 + (7v-6)C \text{ if } v \in \left[\frac{7}{8}, 1\right] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{8v^2}{7} \text{ if } v \in \left[0, \frac{7}{8}\right] \\ 8(v-1)^2 \text{ if } v \in \left[\frac{7}{8}, 1\right] \end{cases} \quad \text{and } P(v) \cdot E_1 = \begin{cases} \frac{8v}{7} \text{ if } v \in \left[0, \frac{7}{8}\right] \\ 8(1-v) \text{ if } v \in \left[\frac{7}{8}, 1\right] \end{cases}$$

We have $S_S(E_1) = \frac{5}{8}$. Thus, $\delta_P(S) \leq \frac{8}{5}$ for $P \in E_1 \backslash E_2$. Moreover, for such points we have

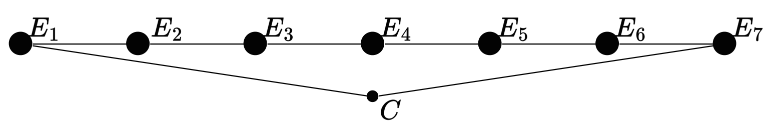
$$h(v) = \begin{cases} \frac{32v^2}{49} \text{ if } v \in \left[0, \frac{7}{8}\right] \\ 8(1-v)(3v-4) \text{ if } v \in \left[\frac{7}{8}, 1\right] \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet}; P) \leq \frac{13}{96} < \frac{5}{8}$. We get $\delta_P(S) = \frac{8}{5}$ for $P \in (E_1 \cup E_7) \backslash (E_2 \cup E_6)$. Thus, $\delta_{\mathcal{P}}(X) = 1$. □

12. $\mathbb{A}_7$ singularity with irreducible ramification divisor on Du Val Del Pezzo surfaces of degree 1

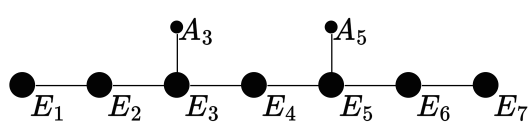
Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_7$ singularity at point $\mathcal{P}$. X can be realized as the double cover $X \xrightarrow{2:1} \mathbb{P}(1,1,2)$, which is ramified along a sextic curve $R \in \mathbb{P}(1,1,2)$. Suppose R is irreducible. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X) = \frac{18}{17}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E_1, E_2, E_3, E_4, E_5, E_6 and E_7 are the exceptional divisors with the intersection:



We have $-K_S \sim C + E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7$. Let P be a point on S. If the ramification divisor R is reducible, then this implies that there is no (-1)-curve that intersects the fundamental cycle only at E_4.

Step 1. Suppose $P \in E_4$. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form the following dual graph:



Then the corresponding Zariski Decomposition of the divisor $-K_S - vE_4$ is:

$$P(v) = \begin{cases} -K_S - vE_4 - \frac{v}{4}(E_1 + 2E_2 + 3E_3 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \left[0, \frac{4}{3}\right] \\ -K_S - vE_4 - (v-1)(E_1 + 2E_2 + 3E_3 + 3E_5 + 2E_6 + E_7) - (3v-4)A_3 \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(E_1 + 2E_2 + 3E_3 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \left[0, \frac{4}{3}\right] \\ (v-1)(E_1 + 2E_2 + 3E_3 + 3E_5 + 2E_6 + E_7) + (3v-4)A_3 \text{ if } v \in \left[\frac{4}{3}, \frac{3}{2}\right] \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_4 \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)E_4 + \frac{1}{2}\Big(E_1 + 2E_2 + 3E_3 + 3E_5 + 2E_6 + E_7 + 2A_3\Big)$$

Moreover,

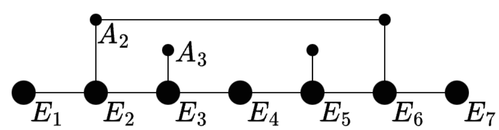
$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{2} \text{ if } v \in \big[0, \frac{4}{3}\big] \\ (3-2v)^2 \text{ if } v \in \big[\frac{4}{3}, \frac{3}{2}\big] \end{cases} \quad \text{and } P(v)\cdot E_4 = \begin{cases} \frac{v}{2} \text{ if } v \in \big[0, \frac{4}{3}\big] \\ 2(3-2v) \text{ if } v \in \big[\frac{4}{3}, \frac{3}{2}\big] \end{cases}$$

We have $S_S(E_4) = \frac{17}{18}$. Thus, $\delta_P(S) \le \frac{18}{17}$ for $P \in E_4$. Moreover, if $P \in E_4$ we have

$$h(v) \le \begin{cases} \frac{v^2}{2} \text{ if } v \in \big[0, \frac{4}{3}\big] \\ 2(3-2v)v \text{ if } v \in \big[\frac{4}{3}, \frac{3}{2}\big] \end{cases}$$

Thus $S(W^{E_2}_{\bullet,\bullet}; P) \le \frac{17}{18}$. We get $\delta_P(S) = \frac{18}{17}$ for $P \in E_4$.

Step 2. Suppose $P \in E_3 \cup E_5$. Without loss of generality we can assume that $P \in E_3$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form the following dual graph:



Then the corresponding Zariski Decomposition of the divisor $-K_S - vE_3$ is:

$$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{15}(5E_1 + 10E_2 + 12E_4 + 9E_5 + 6E_6 + 3E_7) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_3 - \frac{v}{15}(5E_1 + 10E_2 + 12E_4 + 9E_5 + 6E_6 + 3E_7) - (v-1)A_3 \text{ if } v \in \big[1, \frac{3}{2}\big] \\ -K_S - vE_3 - (v-1)(E_1 + 2E_2 + A_3) - \frac{v}{5}(4E_4 + 3E_5 + 2E_6 + E_7) - (2v-3)A_2 \text{ if } v \in \big[\frac{3}{2}, \frac{5}{3}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{15}(5E_1 + 10E_2 + 12E_4 + 9E_5 + 6E_6 + 3E_7) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{15}(5E_1 + 10E_2 + 12E_4 + 9E_5 + 6E_6 + 3E_7) + (v-1)A_3 \text{ if } v \in \big[1, \frac{3}{2}\big] \\ (v-1)(E_1 + 2E_2 + A_3) + \frac{v}{5}(4E_4 + 3E_5 + 2E_6 + E_7) + (2v-3)A_2 \text{ if } v \in \big[\frac{3}{2}, \frac{5}{3}\big] \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_3 \sim_{\mathbb{R}} \Big(\frac{5}{3} - v\Big)E_3 + \frac{1}{3}\Big(2E_1 + 4E_2 + 2A_3 + 4E_4 + 3E_5 + 2E_6 + E_7 + 2A_2\Big)$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{8v^2}{15} \text{ if } v \in \big[0, 1\big] \\ 2 - 2v + \frac{7v^2}{15} \text{ if } v \in \big[1, \frac{3}{2}\big] \\ \frac{(5-3v)^2}{5} \text{ if } v \in \big[\frac{3}{2}, \frac{5}{3}\big] \end{cases} \quad \text{and } P(v)\cdot E_3 = \begin{cases} \frac{8v}{15} \text{ if } v \in \big[0, 1\big] \\ 1 - \frac{7v}{15} \text{ if } v \in \big[1, \frac{3}{2}\big] \\ 3(1 - \frac{3v}{5}) \text{ if } v \in \big[\frac{3}{2}, \frac{5}{3}\big] \end{cases}$$

We have $S_S(E_3) = \frac{17}{18}$. Thus, $\delta_P(S) \le \frac{18}{17}$ for $P \in E_3$. Moreover, if $P \in E_3 \cap A_3$ or if $P \in E_3 \cap E_2$ we have

$$h(v) \le \begin{cases} \frac{32v^2}{225} \text{ if } v \in \big[0, 1\big] \\ \frac{(15-7v)(23v-15)}{450} \text{ if } v \in \big[1, \frac{3}{2}\big] \\ \frac{3(5-3v)(v+5)}{50} \text{ if } v \in \big[\frac{3}{2}, \frac{5}{3}\big] \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{112v^2}{225} \text{ if } v \in \big[0, 1\big] \\ \frac{(15-7v)(13v+15)}{450} \text{ if } v \in \big[1, \frac{3}{2}\big] \\ \frac{3(5-3v)(11v-5)}{50} \text{ if } v \in \big[\frac{3}{2}, \frac{5}{3}\big] \end{cases}$$

Thus $S(W^{E_3}_{\bullet,\bullet};P) \le \frac{14}{45} < \frac{17}{18}$ or $S(W^{E_3}_{\bullet,\bullet};P) \le \frac{37}{45} < \frac{17}{18}$. We get $\delta_P(S) = \frac{18}{17}$ for $P \in (E_3 \cup E_5)\backslash E_4$.
Step 3. Suppose $P \in E_2 \cup E_6$. Without loss of generality we can assume that $P \in E_2$ since the proof is similar in other cases. Then the Zariski Decomposition of the divisor $-K_S - vE_2$ is:

$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{6}(3E_1 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_2 - \frac{v}{6}(3E_1 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) - (v-1)A_2 \text{ if } v \in \big[1, \frac{6}{5}\big] \\ -K_S - vE_2 - \frac{v}{2}E_1 - (v-1)(5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7 + A_2) - (5v-6)A_3 \text{ if } v \in \big[\frac{6}{5}, \frac{4}{3}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{6}(3E_1 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) + (v-1)A_2 \text{ if } v \in \big[1, \frac{6}{5}\big] \\ \frac{v}{2}E_1 + (v-1)(5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7 + A_2) + (5v-6)A_3 \text{ if } v \in \big[\frac{6}{5}, \frac{4}{3}\big] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{2v^2}{3} \text{ if } v \in \big[0, 1\big] \\ 2 - 2v + \frac{v^2}{3} \text{ if } v \in \big[1, \frac{6}{5}\big] \\ \frac{(4-3v)^2}{2} \text{ if } v \in \big[\frac{6}{5}, \frac{4}{3}\big] \end{cases} \quad \text{and } P(v) \cdot E_2 = \begin{cases} \frac{2v}{3} \text{ if } v \in \big[0, 1\big] \\ 1 - \frac{v}{3} \text{ if } v \in \big[1, \frac{6}{5}\big] \\ 3(2 - \frac{3v}{2}) \text{ if } v \in \big[\frac{6}{5}, \frac{4}{3}\big] \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_2 \sim_{\mathbb{R}} \Big(\frac{4}{3} - v\Big)E_2 + \frac{1}{3}\Big(2E_1 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7 + A_2 + 2A_3\Big)$$

We have $S_S(E_2) = \frac{37}{45}$. Thus, $\delta_P(S) \le \frac{45}{37}$ for $P \in E_2$. Moreover, if $P \in E_2 \cap E_1$ or if $P \in E_2\backslash(E_1 \cup E_3)$ we have

$$h(v) \le \begin{cases} \frac{5v^2}{9} \text{ if } v \in \big[0, 1\big] \\ \frac{(3-v)(2v+3)}{18} \text{ if } v \in \big[1, \frac{6}{5}\big] \\ \frac{3(3v-4)(7v-12)}{8} \text{ if } v \in \big[\frac{6}{5}, \frac{4}{3}\big] \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{2v^2}{9} \text{ if } v \in \big[0, 1\big] \\ \frac{(3-v)(5v-3)}{18} \text{ if } v \in \big[1, \frac{6}{5}\big] \\ \frac{3(3v-4)(5v-8)}{8} \text{ if } v \in \big[\frac{6}{5}, \frac{4}{3}\big] \end{cases}$$

Thus, $S(W^{E_2}_{\bullet,\bullet};P) \le \frac{59}{90} < \frac{37}{45}$ or $S(W^{E_2}_{\bullet,\bullet};P) \le \frac{13}{45} < \frac{37}{45}$. We get $\delta_P(S) = \frac{45}{37}$ for $P \in (E_2 \cup E_6)\backslash(E_3 \cup E_5)$.
Step 4. Suppose $P \in E_1 \cup E_7$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. The Zariski decomposition of the divisor $-K_S - vE_1 \sim C + E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7$ is given by:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{7}(6E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \big[0, \frac{7}{8}\big] \\ -K_S - vE_1 - (2v-1)E_2 - (3v-2)E_3 - (4v-3)E_4 - (5v-4)E_5 - (6v-5)E_6 - (7v-6)E_7 - (8v-7)C \text{ if } v \in \big[\frac{7}{8}, 1\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{7}(6E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \big[0, \frac{7}{8}\big] \\ (2v-1)E_2 + (3v-2)E_3 + (4v-3)E_4 + (5v-4)E_5 + (6v-5)E_6 + (7v-6)C \text{ if } v \in \big[\frac{7}{8}, 1\big] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{8v^2}{7} \text{ if } v \in \big[0, \frac{7}{8}\big] \\ 8(v-1)^2 \text{ if } v \in \big[\frac{7}{8}, 1\big] \end{cases} \quad \text{and } P(v) \cdot E_1 = \begin{cases} \frac{8v}{7} \text{ if } v \in \big[0, \frac{7}{8}\big] \\ 8(1-v) \text{ if } v \in \big[\frac{7}{8}, 1\big] \end{cases}$$

We have $S_S(E_1) = \frac{5}{8}$. Thus, $\delta_P(S) \le \frac{8}{5}$ for $P \in E_1\backslash E_2$. Moreover, for such points we have

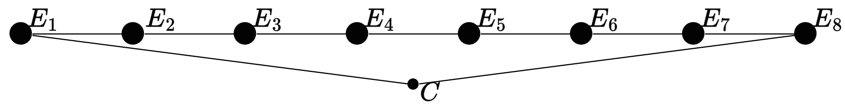
$$h(v) = \begin{cases} \frac{32v^2}{49} \text{ if } v \in \big[0, \frac{7}{8}\big] \\ 8(1-v)(3v-4) \text{ if } v \in \big[\frac{7}{8}, 1\big] \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet};P) \le \frac{13}{96} < \frac{5}{8}$. We get $\delta_P(S) = \frac{8}{5}$ for $P \in (E_1 \cup E_7)\backslash(E_2 \cup E_6)$. Thus, $\delta_{\mathcal{P}}(X) = \frac{18}{17}$. $\square$

13. $\mathbb{A}_8$ singularity on Du Val Del Pezzo surfaces of degree 1

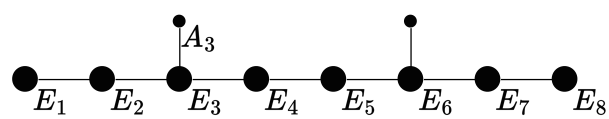
Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{A}_8$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X) = 1$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E_1, E_2, E_3, E_4, E_5, E_6, E_7 and E_8 are the exceptional divisors with the intersection:



We have $-K_S \sim C + E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8$. Let P be a point on S.

Step 1. Suppose $P \in E_4 \cup E_5$. Without loss of generality we can assume that $P \in E_4$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form the following dual graph:



Then the corresponding Zariski Decomposition of the divisor $-K_S - vE_4$ is:

$$P(v) = \begin{cases} -K_S - vE_4 - \frac{v}{20}(5E_1 + 10E_2 + 15E_3 + 16E_5 + 12E_6 + 8E_7 + 4E_8) \text{ if } v \in \big[0, \frac{4}{3}\big] \\ -K_S - vE_4 - (v-1)(E_1 + 2E_2 + 3E_3) - \frac{v}{5}(4E_5 + 3E_6 + 2E_7 + E_8) - (3v-4)A_3 \text{ if } v \in \big[\frac{4}{3}, \frac{5}{3}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{20}(5E_1 + 10E_2 + 15E_3 + 16E_5 + 12E_6 + 8E_7 + 4E_8) \text{ if } v \in \big[0, \frac{4}{3}\big] \\ (v-1)(E_1 + 2E_2 + 3E_3) + \frac{v}{5}(4E_5 + 3E_6 + 2E_7 + E_8) + (3v-4)A_3 \text{ if } v \in \big[\frac{4}{3}, \frac{5}{3}\big] \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_4 \sim_{\mathbb{R}} \Big(\frac{5}{3} - v\Big)E_4 + \frac{1}{3}\Big(2E_1 + 4E_2 + 6E_3 + 4E_5 + 3E_6 + 2E_7 + E_8 + 3A_3\Big)$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{9v^2}{20} \text{ if } v \in \big[0, \frac{4}{3}\big] \\ \frac{(5-3v)^2}{5} \text{ if } v \in \big[\frac{4}{3}, \frac{5}{3}\big] \end{cases} \quad \text{and } P(v) \cdot E_4 = \begin{cases} \frac{9v}{20} \text{ if } v \in \big[0, \frac{4}{3}\big] \\ 3(2 - \frac{3v}{5}) \text{ if } v \in \big[\frac{4}{3}, \frac{5}{3}\big] \end{cases}$$

We have $S_S(E_4) = 1$. Thus, $\delta_P(S) \leq 1$ for $P \in E_4$. Moreover, if $P \in E_4 \cap E_5$ or if $P \in E_4 \backslash E_5$ we have

$$h(v) \leq \begin{cases} \frac{369v^2}{800} \text{ if } v \in \big[0, \frac{4}{3}\big] \\ \frac{3(3v-5)(v-15)}{50} \text{ if } v \in \big[\frac{4}{3}, \frac{5}{3}\big] \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{351v^2}{800} \text{ if } v \in \big[0, \frac{4}{3}\big] \\ \frac{9(3v-5)(5-7v)}{50} \text{ if } v \in \big[\frac{4}{3}, \frac{5}{3}\big] \end{cases}$$

Thus, $S(W^{E_2}_{\bullet,\bullet}; P) \leq 1$. We get $\delta_P(S) = 1$ for $P \in E_4 \cup E_5$.

Step 2. Suppose $P \in E_3 \cup E_5$. Without loss of generality we can assume that $P \in E_3$ since the proof is similar in other cases. The Zariski Decomposition of the divisor $-K_S - vE_3$ is:

$$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{6}(2E_1 + 4E_2 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_8) \text{ if } v \in [0,1] \\ -K_S - vE_3 - \frac{v}{6}(2E_1 + 4E_2 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_8) - (v-1)A_3 \text{ if } v \in [1,2] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(2E_1 + 4E_2 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_8) \text{ if } v \in [0,1] \\ \frac{v}{6}(2E_1 + 4E_2 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_8) + (v-1)A_3 \text{ if } v \in [1,2] \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_3 \sim_{\mathbb{R}} (2-v)E_3 + \frac{1}{3}\Big(2E_1 + 4E_2 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_8\Big) + A_3$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{2} \text{ if } v \in [0,1] \\ \frac{(2-v)^2}{2} \text{ if } v \in [1,2] \end{cases} \quad \text{and } P(v) \cdot E_3 = \begin{cases} \frac{v}{2} \text{ if } v \in [0,1] \\ 1 - \frac{v}{2} \text{ if } v \in [1,2] \end{cases}$$

We have $S_S(E_3) = 1$. Thus, $\delta_P(S) \le 1$ for $P \in E_3$. Moreover, if $P \in E_4 \cap E_2$ or if $P \in E_4 \backslash (E_2 \cup E_4)$ we have

$$h(v) \le \begin{cases} \frac{11v^2}{24} \text{ if } v \in [0,1] \\ \frac{(2-v)(5v+6)}{24} \text{ if } v \in [1,2] \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{v^2}{8} \text{ if } v \in [0,1] \\ \frac{(2-v)(3v-2)}{8} \text{ if } v \in [1,2] \end{cases}$$

Thus, $S(W^{E_3}_{\bullet,\bullet}; P) \le \frac{5}{6} < 1$ or $S(W^{E_3}_{\bullet,\bullet}; P) \le \frac{1}{3} < 1$. We get $\delta_P(S) = 1$ for $P \in (E_3 \cup E_6) \backslash (E_4 \cup E_5)$.

Step 3. Suppose $P \in E_2 \cup E_7$. The Zariski Decomposition of the divisor $-K_S - vE_2$ is:

$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{2}E_1 - \frac{v}{7}(6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_7) \text{ if } v \in [0, \frac{7}{6}] \\ -K_S - vE_2 - \frac{v}{2}E_1 - (v-1)(6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_7) - (6v-7)A_3 \text{ if } v \in [\frac{7}{6}, \frac{4}{3}] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}E_1 + \frac{v}{7}(6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_7) \text{ if } v \in [0, \frac{7}{6}] \\ \frac{v}{2}E_1 + (v-1)(6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_7) + (6v-7)A_3 \text{ if } v \in [\frac{7}{6}, \frac{4}{3}] \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_2 \sim_{\mathbb{R}} \Big(\frac{4}{3} - v\Big)E_2 + \frac{1}{3}\Big(2E_1 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_7 + 3A_3\Big)$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{9v^2}{14} \text{ if } v \in [0, \frac{7}{6}] \\ \frac{(4-3v)^2}{2} \text{ if } v \in [\frac{7}{6}, \frac{4}{3}] \end{cases} \quad \text{and } P(v) \cdot E_2 = \begin{cases} \frac{9v}{14} \text{ if } v \in [0, \frac{7}{6}] \\ 3(1 - \frac{3v}{2}) \text{ if } v \in [\frac{7}{6}, \frac{4}{3}] \end{cases}$$

We have $S_S(E_2) = \frac{5}{6}$. Thus, $\delta_P(S) \le \frac{6}{5}$ for $P \in E_2$. Moreover, if $P \in E_2 \backslash E_3$ we have

$$h(v) \le \begin{cases} \frac{207v^2}{392} \text{ if } v \in [0, \frac{7}{6}] \\ \frac{3(3v-4)(7v-12)}{8} \text{ if } v \in [\frac{7}{6}, \frac{4}{3}] \end{cases}$$

Thus

$$S(W^{E_2}_{\bullet,\bullet}; P) \le 2\Big(\int_0^{7/6} \frac{207v^2}{392} dv + \int_{7/6}^{4/3} \frac{3(3v-4)(7v-12)}{8} dv\Big) = \frac{1}{4} < \frac{5}{6}$$

We get $\delta_P(S) = \frac{6}{5}$ for $P \in (E_2 \cup E_7) \backslash (E_3 \cup E_6)$.

Step 4. Suppose $P \in E_1 \cup E_8$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. The Zariski decomposition of the divisor $-K_S - vE_1 \sim C + (1-v)E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8$ is given by:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{8}(7E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_8) \text{ if } v \in \left[0, \frac{8}{9}\right] \\ -K_S - vE_1 - (2v-1)E_2 - (3v-2)E_3 - (4v-3)E_4 - (5v-4)E_5 - (6v-5)E_6 - (7v-6)E_7 - (8v-7)E_8 - (9v-8)C \text{ if } v \in \left[\frac{8}{9}, 1\right] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{8}(7E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_8) \text{ if } v \in \left[0, \frac{8}{9}\right] \\ (2v-1)E_2 + (3v-2)E_3 + (4v-3)E_4 + (5v-4)E_5 + (6v-5)E_6 + (7v-6)E_8 + (9v-8)C \text{ if } v \in \left[\frac{8}{9}, 1\right] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{9v^2}{8} \text{ if } v \in \left[0, \frac{8}{9}\right] \\ 9(v-1)^2 \text{ if } v \in \left[\frac{8}{9}, 1\right] \end{cases} \quad \text{and } P(v) \cdot E_1 = \begin{cases} \frac{9v}{8} \text{ if } v \in \left[0, \frac{8}{9}\right] \\ 9(1-v) \text{ if } v \in \left[\frac{8}{9}, 1\right] \end{cases}$$

We have $S_S(E_1) = \frac{17}{27}$. Thus, $\delta_P(S) \leq \frac{27}{17}$ for $P \in E_1 \backslash E_2$. Moreover, for such points we have

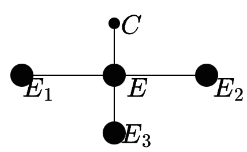
$$h(v) = \begin{cases} \frac{81v^2}{128} \text{ if } v \in \left[0, \frac{8}{9}\right] \\ \frac{9(1-v)(9v-7)}{2} \text{ if } v \in \left[\frac{8}{9}, 1\right] \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet}; P) \leq \frac{10}{27} < \frac{17}{27}$. We get $\delta_P(S) = \frac{27}{17}$ for $P \in (E_1 \cup E_8) \backslash (E_2 \cup E_7)$. Thus, $\delta_{\mathcal{P}}(X) = 1$. ☐

14. $\mathbb{D}_4$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{D}_4$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X) = 1$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E, E_1, E_2 and E_3 are the exceptional divisors with the intersection:



We have $-K_S \sim C + 2E + E_1 + E_2 + E_3$. Let P be a point on S.
Step 1. Suppose $P \in E$. The Zariski decomposition of the divisor $-K_S - vE \sim (2-v)E + E_1 + E_2 + E_3 + C$ is:

$$P(v) = \begin{cases} -K_S - vE - \frac{v}{2}(E_1 + E_2 + E_3) \text{ if } v \in \left[0, 1\right] \\ -K_S - vE - \frac{v}{2}(E_1 + E_2 + E_3) - (v-1)C \text{ if } v \in \left[1, 2\right] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(E_1 + E_2 + E_3) \text{ if } v \in \left[0, 1\right] \\ \frac{v}{2}(E_1 + E_2 + E_3) + (v-1)C \text{ if } v \in \left[1, 2\right] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{2} \text{ if } v \in \left[0, 1\right] \\ \frac{(2-v)^2}{2} \text{ if } v \in \left[1, 2\right] \end{cases} \quad \text{and } P(v) \cdot E = \begin{cases} \frac{v}{2} \text{ if } v \in \left[0, 1\right] \\ 1 - \frac{v}{2} \text{ if } v \in \left[1, 2\right] \end{cases}$$

We have $S_S(E) = 1$ Thus, $\delta_P(S) \leq 1$ for $P \in E$. Moreover, if $P \in E \cap (E_1 \cup E_2 \cup E_3)$ or if $P \in E\backslash(E_1 \cup E_2 \cup E_3)$ we have

$$h(v) \leq \begin{cases} \frac{3v^2}{8} \text{ if } v \in \big[0,1\big] \\ \frac{(2-v)(2+v)}{24} \text{ if } v \in \big[1,2\big] \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{v^2}{8} \text{ if } v \in \big[0,1\big] \\ \frac{(2-v)(3v-2)}{8} \text{ if } v \in \big[1,2\big] \end{cases}$$

Thus, $S(W^E_{\bullet,\bullet}; P) \leq \frac{2}{3} < 1$ or $S(W^E_{\bullet,\bullet}; P) \leq \frac{1}{3} < 1$. We get $\delta_P(S) = 1$ for $P \in E$.

Step 2. Suppose $P \in E_1 \cup E_2 \cup E_3$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. The Zariski decomposition of the divisor $-K_S - vE_1 \sim C + 2E + (1-v)E_1 + E_2 + E_3$ is given by:

$$P(v) = -K_S - vE_1 - \frac{v}{2}(2E + E_1 + E_2) \text{ and } N(v) = \frac{v}{2}(2E + E_1 + E_2) \text{ if } v \in \big[0,1\big]$$

Moreover,

$$(P(v))^2 = (1-v)(1+v) \text{ and } P(v) \cdot E_1 = v \text{ if } v \in \big[0,1\big]$$

We have $S_S(E_1) = \frac{2}{3}$. Thus, $\delta_P(S) \leq \frac{3}{2}$ for $P \in E_1$. Moreover, for $E_1\backslash E$ such points we have

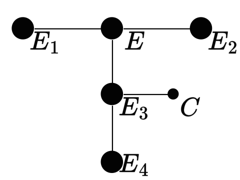
$$h(v) \leq \frac{v^2}{2} \text{ if } v \in \big[0,1\big]$$

Thus, $S(W^{E_1}_{\bullet,\bullet}; P) \leq \frac{1}{3} < \frac{2}{3}$. We get $\delta_P(S) = \frac{3}{2}$ for $P \in (E_1 \cup E_2 \cup E_3)\backslash E$. Thus, $\delta_{\mathcal{P}}(X) = 1$. □

15. $\mathbb{D}_5$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{D}_5$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X) = \frac{6}{7}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E, E_1, E_2, E_3 and E_4 are the exceptional divisors with the intersection:



We have $-K_S \sim C + E_1 + E_2 + 2E + 2E_3 + E_4$. Let P be a point on S.

Step 1. Suppose $P \in E$. The Zariski decomposition of the divisor $-K_S - vE \sim (2-v)E + E_1 + E_2 + 2E_3 + E_4 + C$ is:

$$P(v) = \begin{cases} -K_S - vE - \frac{v}{6}(3E_1 + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in \big[0, \frac{3}{2}\big] \\ -K_S - vE - \frac{v}{2}(E_1 + E_2) - (v-1)(2E_3 + E_4) - (2v-3)C \text{ if } v \in \big[\frac{3}{2}, 2\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in \big[0, \frac{3}{2}\big] \\ \frac{v}{2}(E_1 + E_2) + (v-1)(2E_3 + E_4) + (2v-3)C \text{ if } v \in \big[\frac{3}{2}, 2\big] \end{cases}$$

Moreover,

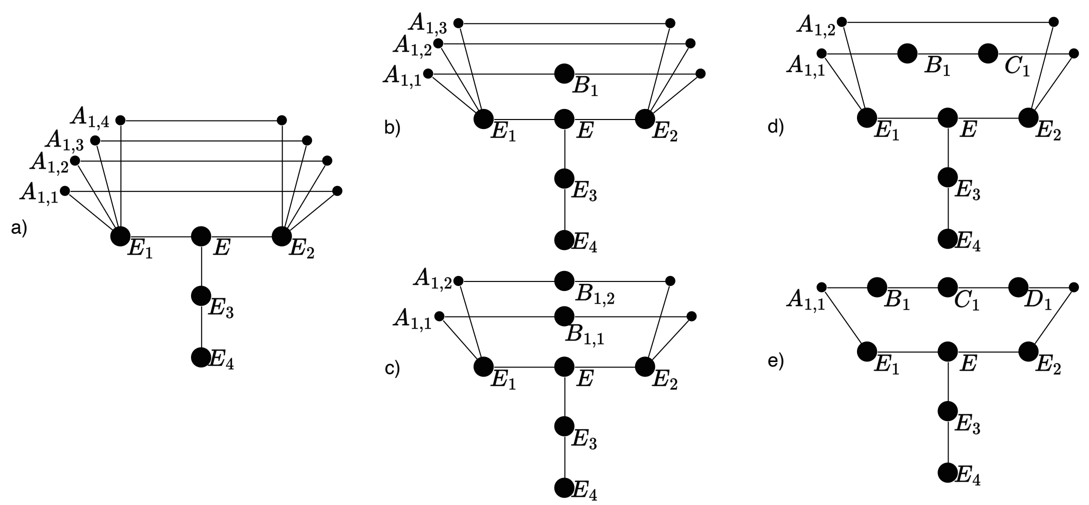
$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{3} \text{ if } v \in \big[0, \frac{3}{2}\big] \\ (2-v)^2 \text{ if } v \in \big[\frac{3}{2}, 2\big] \end{cases} \quad \text{and } P(v) \cdot E = \begin{cases} \frac{v}{3} \text{ if } v \in \big[0, \frac{3}{2}\big] \\ 2 - v \text{ if } v \in \big[\frac{3}{2}, 2\big] \end{cases}$$

We have $S_S(E) = \frac{7}{6}$. Thus, $\delta_P(S) \le \frac{6}{7}$ for $P \in E$. Moreover, if $P \in E \cap (E_1 \cup E_2)$ or if $P \in E \backslash (E_1 \cup E_2)$ we have

$$h(v) \le \begin{cases} \frac{2v^2}{9} \text{ if } v \in \big[0, \frac{3}{2}\big] \\ 2 - v \text{ if } v \in \big[\frac{3}{2}, 2\big] \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{5v^2}{18} \text{ if } v \in \big[0, \frac{3}{2}\big] \\ \frac{(2-v)(3v-2)}{2} \text{ if } v \in \big[\frac{3}{2}, 2\big] \end{cases}$$

Thus, $S(W^E_{\bullet,\bullet}; P) \le \frac{3}{4} < \frac{7}{6}$ or $S(W^E_{\bullet,\bullet}; P) \le 1 < \frac{7}{6}$. We get $\delta_P(S) = \frac{6}{7}$ for $P \in E$.

Step 2. Suppose $P \in E_1 \cup E_2$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:



Then the corresponding Zariski Decomposition of the divisor $-K_S - vE_1$ is:

a).
$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_1 - \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) - (v-1)(A_{1,1} + A_{1,2} + A_{1,3} + A_{1,4}) \text{ if } v \in \big[1, \frac{5}{4}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) + (v-1)(A_{1,1} + A_{1,2} + A_{1,3} + A_{1,4}) \text{ if } v \in \big[1, \frac{5}{4}\big] \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_1 - \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) - (v-1)(2A_{1,1} + B_1 + A_{1,2} + A_{1,3}) \text{ if } v \in \big[1, \frac{5}{4}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) + (v-1)(2A_{1,1} + B_1 + A_{1,2} + A_{1,3}) \text{ if } v \in \big[1, \frac{5}{4}\big] \end{cases}$$

c).
$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_1 - \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) - (v-1)(2A_{1,1} + B_{1,1} + A_{1,2} + B_{1,2}) \text{ if } v \in \big[1, \frac{5}{4}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) + (v-1)(2A_{1,1} + B_{1,1} + A_{1,2} + B_{1,2}) \text{ if } v \in \big[1, \frac{5}{4}\big] \end{cases}$$

d).
$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_1 - \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) - (v-1)(3A_{1,1} + 2B_1 + C_1 + A_{1,2}) \text{ if } v \in \big[1, \frac{5}{4}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{5}(6E + 3E_2 + 4E_3 + 2E_4) + (v-1)(3A_{1,1} + 2B_1 + C_1 + A_{1,2}) \text{ if } v \in \big[1, \frac{5}{4}\big] \end{cases}$$

e). $$P(v)=\begin{cases}-K_S-vE_1-\frac{v}{5}(6E+3E_2+4E_3+2E_4) \text{ if } v\in\big[0,1\big]\\ -K_S-vE_1-\frac{v}{5}(6E+3E_2+4E_3+2E_4)-(v-1)(4A_{1,1}+3B_1+2C_1+D_1) \text{ if } v\in\big[1,\frac{5}{4}\big]\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{5}(6E+3E_2+4E_3+2E_4) \text{ if } v\in\big[0,1\big]\\ \frac{v}{5}(6E+3E_2+4E_3+2E_4)+(v-1)(4A_{1,1}+3B_1+2C_1+D_1) \text{ if } v\in\big[1,\frac{5}{4}\big]\end{cases}$$

The Zariski Decomposition in part a). follows from

$$-K_S-vE_1\sim_{\mathbb{R}}\Big(\frac{5}{4}-v\Big)E_1+\frac{1}{4}\Big(6E+3E_2+4E_3+2E_4+A_{1,1}+A_{1,2}+A_{1,3}+A_{1,4}\Big)$$

A similar statement holds in other parts. Moreover,

$$(P(v))^2=\begin{cases}1-\frac{4v^2}{5} \text{ if } v\in\big[0,1\big]\\ \frac{(5-4v)^2}{5} \text{ if } v\in\big[1,\frac{5}{4}\big]\end{cases} \quad \text{and } P(v)\cdot E_1=\begin{cases}\frac{4v}{5} \text{ if } v\in\big[0,1\big]\\ 4(1-\frac{4v}{5}) \text{ if } v\in\big[1,\frac{5}{4}\big]\end{cases}$$

We have $S_S(E_1)=\frac{3}{4}$. Thus, $\delta_P(S)\le\frac{4}{3}$ for $P\in E_1$. Moreover, if $P\in E_1\backslash E$ we have

$$h(v)\le\begin{cases}\frac{8v^2}{25} \text{ if } v\in\big[0,1\big]\\ \frac{4(5-4v)(7v-5)}{25} \text{ if } v\in\big[1,\frac{5}{4}\big]\end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet};P)\le\frac{19}{60}<\frac{3}{4}$. We get $\delta_P(S)=\frac{4}{3}$ for $P\in(E_1\cup E_2)\backslash E$.

Step 3. Suppose $P\in E_3$. The Zariski decomposition of the divisor $-K_S-vE_3\sim C+E_1+E_2+2E+(2-v)E_3+E_4$ is:

$$P(v)=\begin{cases}-K_S-vE_3-\frac{v}{2}(E_4+2E+E_1+E_2) \text{ if } v\in\big[0,1\big]\\ -K_S-vE_3-\frac{v}{2}(E_4+2E+E_1+E_2)-(v-1)C \text{ if } v\in\big[1,2\big]\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{2}(E_4+2E+E_1+E_2) \text{ if } v\in\big[0,1\big]\\ \frac{v}{2}(E_4+2E+E_1+E_2)+(v-1)C \text{ if } v\in\big[1,2\big]\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{2} \text{ if } v\in\big[0,1\big]\\ \frac{(2-v)^2}{2} \text{ if } v\in\big[1,2\big]\end{cases} \quad \text{and } P(v)\cdot E_3=\begin{cases}\frac{v}{2} \text{ if } v\in\big[0,1\big]\\ 1-\frac{v}{2} \text{ if } v\in\big[1,2\big]\end{cases}$$

Now we apply the computation from Section 14 (Step 1.) and get that $\delta_P(S)=1$ for $P\in E_3\backslash E$.

Step 4. Suppose $P\in E_4$. The Zariski decomposition of the divisor $-K_S-vE_4\sim C+E_1+E_2+2E+2E_3+(1-v)E_4$ is given by:

$$P(v)=-K_S-vE_4-\frac{v}{2}(2E_3+2E+E_1+E_2) \text{ and } N(v)=\frac{v}{2}(2E_3+2E+E_1+E_2) \text{ if } v\in\big[0,1\big]$$

Moreover,

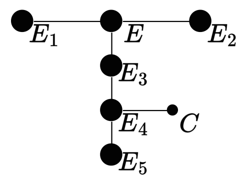
$$(P(v))^2=(1-v)(1+v) \text{ and } P(v)\cdot E_4=v \text{ if } v\in\big[0,1\big]$$

Now we apply the computation from Section 14 (Step 2.) and get that $\delta_P(S)=\frac{3}{2}$ for $P\in E_4\backslash E_3$. Thus, $\delta_{\mathcal{P}}(X)=\frac{6}{7}$. □

16. $\mathbb{D}_6$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{D}_6$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X) = \frac{3}{4}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E, E_1, E_2, E_3, E_4 and E_5 are the exceptional divisors with the intersection:



We have $-K_S \sim C + E_1 + E_2 + 2E + 2E_3 + 2E_4 + E_5$. Let P be a point on S.
Step 1. Suppose $P \in E$. The Zariski decomposition of the divisor $-K_S - vE \sim C + E_1 + E_2 + (2 - v)E + 2E_3 + 2E_4 + E_5$ is given by:

$$P(v) = -K_S - vE - \frac{v}{4}(2E_1 + 2E_2 + 3E_3 + 2E_4 + E_5) \text{ and } N(v) = \frac{v}{4}(2E_1 + 2E_2 + 3E_3 + 2E_4 + E_5) \text{ if } v \in \big[0, 2\big]$$

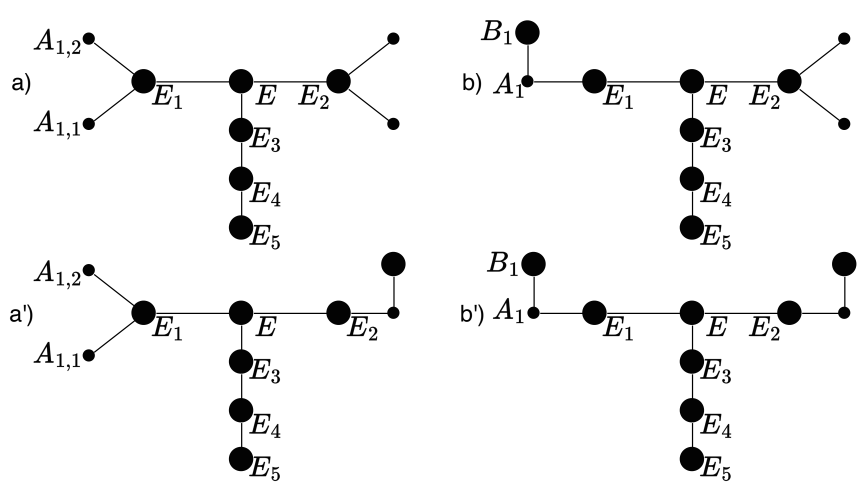
Moreover,

$$(P(v))^2 = \frac{(2 - v)(2 + v)}{4} \text{ and } P(v) \cdot E = \frac{v}{4} \text{ if } v \in \big[0, 2\big]$$

We have $S_S(E) = \frac{4}{3}$. Thus, $\delta_P(S) \leq \frac{3}{4}$ for $P \in E$. Moreover, for such points we have

$$h(v) \leq \frac{7v^2}{32} \text{ if } v \in \big[0, 2\big]$$

Thus, $S(W^{E}_{\bullet,\bullet}; P) \leq \frac{7}{6} < \frac{4}{3}$. We get $\delta_P(S) = \frac{3}{4}$ for $P \in E$.
Step 2. Suppose $P \in E_1 \cup E_2$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:



Then the corresponding Zariski Decomposition of the divisor $-K_S - vE_1$ is:

a, a'). $$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{3}(4E + 2E_2 + 3E_3 + 2E_4 + E_5) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_1 - \frac{v}{3}(4E + 2E_2 + 3E_3 + 2E_4 + E_5) - (v-1)(A_{1,1} + A_{1,2}) \text{ if } v \in \big[1, \frac{3}{2}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(4E + 2E_2 + 3E_3 + 2E_4 + E_5) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{3}(4E + 2E_2 + 3E_3 + 2E_4 + E_5) + (v-1)(A_{1,1} + A_{1,2}) \text{ if } v \in \big[1, \frac{3}{2}\big] \end{cases}$$

b, b'). $$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{3}(4E + 2E_2 + 3E_3 + 2E_4 + E_5) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_1 - \frac{v}{3}(4E + 2E_2 + 3E_3 + 2E_4 + E_5) - (v-1)(2A_{1,1} + B_1) \text{ if } v \in \big[1, \frac{3}{2}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(4E + 2E_2 + 3E_3 + 2E_4 + E_5) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{3}(4E + 2E_2 + 3E_3 + 2E_4 + E_5) + (v-1)(2A_{1,1} + B_1) \text{ if } v \in \big[1, \frac{3}{2}\big] \end{cases}$$

The Zariski Decomposition in part a). follows from

$$-K_S - vE_1 \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)E_1 + \frac{1}{2}\Big(4E + 2E_2 + 3E_3 + 2E_4 + E_5 + A_{1,1} + A_{1,2}\Big)$$

A similar statement holds in other parts. Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{2v^2}{3} \text{ if } v \in \big[0, 1\big] \\ \frac{(3-2v)^2}{3} \text{ if } v \in \big[1, \frac{3}{2}\big] \end{cases} \quad \text{and } P(v) \cdot E_1 = \begin{cases} \frac{2v}{3} \text{ if } v \in \big[0, 1\big] \\ 2(1 - \frac{2v}{3}) \text{ if } v \in \big[1, \frac{3}{2}\big] \end{cases}$$

We have $S_S(E_1) = \frac{5}{6}$. Thus, $\delta_P(S) \le \frac{6}{5}$ for $P \in E_1$. Moreover, if $P \in E_1 \backslash E$ we have

$$h(v) \le \begin{cases} \frac{2v^2}{9} \text{ if } v \in \big[0, 1\big] \\ \frac{2(2v-3)(4v-3)}{9} \text{ if } v \in \big[1, \frac{3}{2}\big] \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet}; P) \le \frac{1}{3} < \frac{5}{6}$. We get $\delta_P(S) = \frac{6}{5}$ for $P \in (E_1 \cup E_2) \backslash E$.

Step 3. Suppose $P \in E_3$. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities where the configuration of all the (-1) and (-2) curves is known so the Zariski decomposition of the divisor $-K_S - vE_3 \sim C + E_1 + E_2 + 2E + (2-v)E_3 + 2E_4 + E_5$ is:

$$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{6}(3E_1 + 3E_2 + 6E + 4E_4 + 2E_5) \text{ if } v \in \big[0, \frac{3}{2}\big] \\ -K_S - vE_3 - \frac{v}{2}(2E + E_1 + E_2) - (v-1)(2E_4 + E_5) - (2v-3)C \text{ if } v \in \big[\frac{3}{2}, 2\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 3E_2 + 6E + 4E_4 + 2E_5) \text{ if } v \in \big[0, \frac{3}{2}\big] \\ \frac{v}{2}(2E + E_1 + E_2) + (v-1)(2E_4 + E_5) + (2v-3)C \text{ if } v \in \big[\frac{3}{2}, 2\big] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{3} \text{ if } v \in \big[0, \frac{3}{2}\big] \\ (2-v)^2 \text{ if } v \in \big[\frac{3}{2}, 2\big] \end{cases} \quad \text{and } P(v) \cdot E_3 = \begin{cases} \frac{v}{3} \text{ if } v \in \big[0, \frac{3}{2}\big] \\ 2 - v \text{ if } v \in \big[\frac{3}{2}, 2\big] \end{cases}$$

Now we apply the computation from Section 15 (Step 1.) and get that $\delta_P(S) = \frac{6}{7}$ for $P \in E_3 \backslash E$.

Step 4. Suppose $P \in E_4$. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities where the configuration of all the (-1) and (-2) curves is known so the Zariski decomposition of the divisor $-K_S - vE_4 \sim$

$C+E_1+E_2+2E+2E_3+(2-v)E_4+E_5$ is:

$$P(v)=\begin{cases}-K_S-vE_4-\frac{v}{2}(2E_3+2E+E_1+E_2+E_5) \text{ if } v\in\big[0,1\big]\\ -K_S-vE_4-\frac{v}{2}(2E_3+2E+E_1+E_2+E_5)-(v-1)C \text{ if } v\in\big[1,2\big]\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{2}(2E_3+2E+E_1+E_2+E_5) \text{ if } v\in\big[0,1\big]\\ \frac{v}{2}(2E_3+2E+E_1+E_2+E_5)+(v-1)C \text{ if } v\in\big[1,2\big]\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{2} \text{ if } v\in\big[0,1\big]\\ \frac{(2-v)^2}{2} \text{ if } v\in\big[1,2\big]\end{cases} \quad\text{and } P(v)\cdot E_4=\begin{cases}\frac{v}{2} \text{ if } v\in\big[0,1\big]\\ 1-\frac{v}{2} \text{ if } v\in\big[1,2\big]\end{cases}$$

Now we apply the computation from Section 14 (Step 1.) and get $\delta_P(S)=1$ for $P\in E_4\backslash E_3$.

Step 5. Suppose $P\in E_5$. The Zariski decomposition of the divisor $-K_S-vE_5\sim C+E_1+E_2+2E+2E_3+2E_4+(1-v)E_5$ is given by:

$$P(v)=-K_S-vE_5-\frac{v}{2}(2E_4+2E_3+2E+E_1+E_2) \text{ and } N(v)=\frac{v}{2}(2E_4+2E_3+2E+E_1+E_2) \text{ if } v\in\big[0,1\big]$$

Moreover,

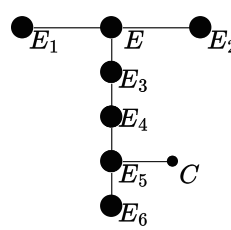
$$(P(v))^2=(1-v)(1+v) \text{ and } P(v)\cdot E_5=v \text{ if } v\in\big[0,1\big]$$

Now we apply the computation from Section 14 (Step 2.) and get that $\delta_P(S)=\frac{3}{2}$ for $P\in E_5\backslash E_4$. Thus, $\delta_{\mathcal{P}}(X)=\frac{3}{4}$. □

17. $\mathbb{D}_7$ singularity on Du Val Del Pezzo surfaces of degree 1

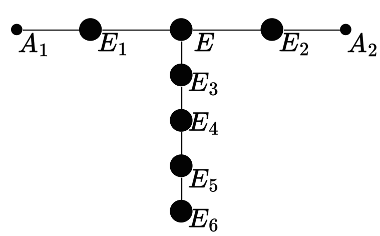
Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{D}_7$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X)=\frac{2}{3}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E, E_1, E_2, E_3, E_4, E_5 and E_6 are the exceptional divisors with the intersection:



We have $-K_S\sim C+E_1+E_2+2E+2E_3+2E_4+2E_5+E_6$. Let P be a point on S.

Step 1. Suppose $P\in E$. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form the following dual graph:



Then the corresponding Zariski Decomposition of the divisor $-K_S - vE$ is:

$$P(v) = \begin{cases} -K_S - vE - \frac{v}{2}(E_1 + E_2) - \frac{v}{5}(4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in \big[0, 2\big] \\ -K_S - vE - (v-1)(E_1 + E_2) - \frac{v}{5}(4E_3 + 3E_4 + 2E_5 + E_6) - (v-2)(A_1 + A_2) \text{ if } v \in \big[2, \frac{5}{2}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(E_1 + E_2) + \frac{v}{5}(4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in \big[0, 2\big] \\ (v-1)(E_1 + E_2) + \frac{v}{5}(4E_3 + 3E_4 + 2E_5 + E_6) + (v-2)(A_1 + A_2) \text{ if } v \in \big[2, \frac{5}{2}\big] \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE \sim_{\mathbb{R}} \Big(\frac{5}{2} - v\Big)E + \frac{1}{2}\Big(3E_1 + 3E_2 + 4E_3 + 3E_4 + 2E_5 + E_6 + A_1 + A_2\Big)$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{5} \text{ if } v \in \big[0, 2\big] \\ \frac{(5-2v)^2}{5} \text{ if } v \in \big[2, \frac{5}{2}\big] \end{cases} \quad \text{and } P(v) \cdot E = \begin{cases} \frac{v}{5} \text{ if } v \in \big[0, 2\big] \\ 2(1 - \frac{2v}{5}) \text{ if } v \in \big[2, \frac{5}{2}\big] \end{cases}$$

We have $S_S(E) = \frac{3}{2}$. Thus, $\delta_P(S) \le \frac{2}{3}$ for $P \in E$. Moreover, if $P \in E \cap E_3$ if $P \in E \backslash E_3$ we have

$$h(v) \le \begin{cases} \frac{9v^2}{50} \text{ if } v \in \big[0, 2\big] \\ \frac{2(5-2v)(2v+5)}{25} \text{ if } v \in \big[2, \frac{5}{2}\big] \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{3v^2}{25} \text{ if } v \in \big[0, 2\big] \\ \frac{6v(5-2v)}{25} \text{ if } v \in \big[2, \frac{5}{2}\big] \end{cases}$$

Thus $S(W^E_{\bullet,\bullet}; P) \le \frac{4}{3} < \frac{3}{2}$ or $S(W^E_{\bullet,\bullet}; P) \le \frac{9}{10} < \frac{3}{2}$. We get $\delta_P(S) = \frac{2}{3}$ for $P \in E$.

Step 2. Suppose $P \in E_1 \cup E_2$. The Zariski decomposition of the divisor $-K_S - vE_3$ is the following:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{7}(10E + 5E_2 + 8E_3 + 6E_4 + 4E_5 + 2E_6) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_1 - \frac{v}{7}(10E + 5E_2 + 8E_3 + 6E_4 + 4E_5 + 2E_6) - (v-1)A_1 \text{ if } v \in \big[1, \frac{7}{5}\big] \\ -K_S - vE_1 - (v-1)(10E + 8E_3 + 6E_4 + 4E_5 + 2E_6 + A_1) - (5v-6)E_2 - (5v-7)A_2 \text{ if } v \in \big[\frac{7}{5}, \frac{3}{2}\big] \end{cases}$$

and

$$N(v) = \begin{cases} \frac{v}{7}(10E + 5E_2 + 8E_3 + 6E_4 + 4E_5 + 2E_6) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{7}(10E + 5E_2 + 8E_3 + 6E_4 + 4E_5 + 2E_6) + (v-1)A_1 \text{ if } v \in \big[1, \frac{7}{5}\big] \\ (v-1)(10E + 8E_3 + 6E_4 + 4E_5 + 2E_6 + A_1) + (5v-6)E_2 + (5v-7)A_2 \text{ if } v \in \big[\frac{7}{5}, \frac{3}{2}\big] \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_1 \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)E_1 + \frac{1}{2}\Big(2A_2 + 3E_2 + 5E + 4E_3 + 3E_4 + 2E_5 + E_6 + A_1\Big)$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{4v^2}{7} \text{ if } v \in \big[0, 1\big] \\ 2 - 2v + \frac{3v^2}{7} \text{ if } v \in \big[1, \frac{7}{5}\big] \\ (3-2v)^2 \text{ if } v \in \big[\frac{7}{5}, \frac{3}{2}\big] \end{cases} \quad \text{and } P(v) \cdot E_1 = \begin{cases} \frac{4v}{7} \text{ if } v \in \big[0, 1\big] \\ 1 - \frac{3v}{7} \text{ if } v \in \big[1, \frac{7}{5}\big] \\ 2(3-2v) \text{ if } v \in \big[\frac{7}{5}, \frac{3}{2}\big] \end{cases}$$

We have $S_S(E_3) = \frac{9}{10}$. Thus, $\delta_P(S) \le \frac{10}{9}$ for $P \in E_1$. Moreover, if $P \in E_1 \backslash E$ we have

$$h(v) \le \begin{cases} \frac{8v^2}{49} \text{ if } v \in \big[0, 1\big] \\ \frac{(7-3v)(11v-7)}{98} \text{ if } v \in \big[1, \frac{7}{5}\big] \\ 2(3-2v)(2-v) \text{ if } v \in \big[\frac{7}{5}, \frac{3}{2}\big] \end{cases}$$

Thus, $S(W^{E_1}_{\bullet,\bullet}; P) \le \frac{3}{10} < \frac{9}{10}$. We get $\delta_P(S) = \frac{10}{9}$ for $P \in (E_1 \cup E_2) \backslash E$.

Step 3. Suppose $P \in E_3$. The Zariski decomposition of the divisor $-K_S - vE_3 \sim C + E_1 + E_2 +$

$2E+(2-v)E_3+2E_4+2E_5+E_6$ is given by:

$$P(v)=-K_S-vE_3-\frac{v}{4}(2E_1+2E_2+4E+3E_4+2E_5+E_6) \text{ if } v\in\big[0,2\big]$$
$$N(v)=\frac{v}{4}(2E_1+2E_2+4E+3E_4+2E_5+E_6) \text{ if } v\in\big[0,2\big]$$

Moreover,

$$(P(v))^2=\frac{(2-v)(2+v)}{4} \text{ and } P(v)\cdot E_3=\frac{v}{4} \text{ if } v\in\big[0,2\big]$$

Now we apply the computation from Section 16 (Step 1.) and get that $\delta_P(S)=\frac{3}{4}$ for $P\in E_3\backslash E$.

Step 4. Suppose $P\in E_4$. The Zariski decomposition of the divisor $-K_S-vE_4\sim C+E_1+E_2+2E+2E_3+(2-v)E_4+2E_5+E_6$ is:

$$P(v)=\begin{cases}-K_S-vE_4-\frac{v}{6}(3E_1+3E_2+6E+6E_3+4E_5+2E_6) \text{ if } v\in\big[0,\frac{3}{2}\big]\\ -K_S-vE_4-\frac{v}{2}(E_1+E_2+2E+2E_3)-(v-1)(2E_5+E_6)-(2v-3)C \text{ if } v\in\big[\frac{3}{2},2\big]\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{6}(3E_1+3E_2+6E+6E_3+4E_5+2E_6) \text{ if } v\in\big[0,\frac{3}{2}\big]\\ \frac{v}{2}(E_1+E_2+2E+2E_3)+(v-1)(2E_5+E_6)+(2v-3)C \text{ if } v\in\big[\frac{3}{2},2\big]\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{3} \text{ if } v\in\big[0,\frac{3}{2}\big]\\ (2-v)^2 \text{ if } v\in\big[\frac{3}{2},2\big]\end{cases} \quad \text{ and } P(v)\cdot E_4=\begin{cases}\frac{v}{3} \text{ if } v\in\big[0,\frac{3}{2}\big]\\ 2-v \text{ if } v\in\big[\frac{3}{2},2\big]\end{cases}$$

Now we apply the computation from Section 15 (Step 1.) and get that $\delta_P(S)=\frac{6}{7}$ for $P\in E_4\backslash E_3$.

Step 5. Suppose $P\in E_5$. The Zariski decomposition of the divisor $-K_S-vE_5\sim C+E_1+E_2+2E+2E_3+2E_4+(2-v)E_5+E_6$ is:

$$P(v)=\begin{cases}-K_S-vE_5-\frac{v}{2}(2E_4+2E_3+2E+E_1+E_2+E_6) \text{ if } v\in\big[0,1\big]\\ -K_S-vE_5-\frac{v}{2}(2E_4+2E_3+2E+E_1+E_2+E_6)-(v-1)C \text{ if } v\in\big[1,2\big]\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}(2E_4+2E_3+2E+E_1+E_2+E_6) \text{ if } v\in\big[0,1\big]\\ \frac{v}{2}(2E_4+2E_3+2E+E_1+E_2+E_6)+(v-1)C \text{ if } v\in\big[1,2\big]\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{2} \text{ if } v\in\big[0,1\big]\\ \frac{(2-v)^2}{2} \text{ if } v\in\big[1,2\big]\end{cases} \quad \text{ and } P(v)\cdot E_5=\begin{cases}\frac{v}{2} \text{ if } v\in\big[0,1\big]\\ 1-\frac{v}{2} \text{ if } v\in\big[1,2\big]\end{cases}$$

Now we apply the computation from Section 14 (Step 1.) and get that $\delta_P(S)=1$ for $P\in E_5\backslash E_4$.

Step 6. Suppose $P\in E_6$. The Zariski decomposition of the divisor $-K_S-vE_6\sim C+E_1+E_2+2E+2E_3+2E_4+2E_5+(1-v)E_6$ is given by:

$$P(v)=-K_S-vE_6-\frac{v}{2}(2E_5+2E_4+2E_3+2E+E_1+E_2) \text{ if } v\in\big[0,1\big]$$
$$N(v)=\frac{v}{2}(2E_5+2E_4+2E_3+2E+E_1+E_2) \text{ if } v\in\big[0,1\big]$$

Moreover,

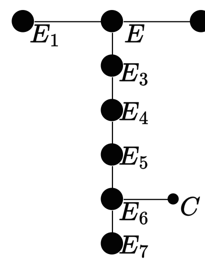
$$(P(v))^2=(1-v)(1+v) \text{ and } P(v)\cdot E_6=v \text{ if } v\in\big[0,1\big]$$

Now we apply the computation from Section 14 (Step 2.) and get that $\delta_P(S)=\frac{3}{2}$ for $P\in E_6\backslash E_5$. Thus, $\delta_{\mathcal{P}}(X)=\frac{2}{3}$. □

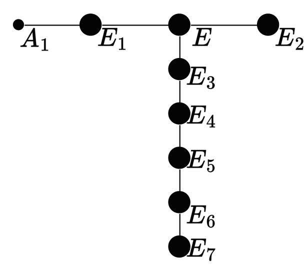
18. $\mathbb{D}_8$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{D}_8$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X) = \frac{3}{5}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E, E_1, E_2, E_3, E_4, E_5, E_6 and E_7 are the exceptional divisors with the intersection:



We have $-K_S \sim C + E_1 + E_2 + 2E + 2E_3 + 2E_4 + 2E_5 + 2E_6 + E_7$. Let P be a point on S.
Step 1. Suppose $P \in E$.If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form the following dual graph:



Then the corresponding Zariski Decomposition of the divisor $-K_S - vE$ is:

$$P(v) = \begin{cases} -K_S - vE - \frac{v}{6}(3E_1 + 3E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \big[0, 2\big] \\ -K_S - vE - (v-1)E_1 - \frac{v}{6}(3E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) - (v-2)A_1 \text{ if } v \in \big[2, 3\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 3E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \big[0, 2\big] \\ (v-1)E_1 + \frac{v}{6}(3E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) + (v-2)A_1 \text{ if } v \in \big[2, 3\big] \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE \sim_{\mathbb{R}} (3-v)E + \frac{1}{2}\Big(4E_1 + 3E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7 + 2A_1\Big)$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{6} \text{ if } v \in \big[0, 2\big] \\ \frac{(3-v)^2}{3} \text{ if } v \in \big[2, 3\big] \end{cases} \quad \text{and } P(v) \cdot E = \begin{cases} \frac{v}{6} \text{ if } v \in \big[0, 2\big] \\ 1 - \frac{v}{6} \text{ if } v \in \big[2, 3\big] \end{cases}$$

We have $S_S(E) = \frac{5}{3}$. Thus, $\delta_P(S) \leq \frac{3}{5}$ for $P \in E$. Moreover, if $P \in E \cap E_1$ if $P \in E \backslash E_1$ we have

$$h(v) \leq \begin{cases} \frac{7v^2}{72} \text{ if } v \in \big[0, 2\big] \\ \frac{(3-v)(5v-3)}{18} \text{ if } v \in \big[2, 3\big] \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{11v^2}{72} \text{ if } v \in \big[0, 2\big] \\ \frac{(3-v)(4v+3)}{18} \text{ if } v \in \big[2, 3\big] \end{cases}$$

Thus, $S(W^{E}_{\bullet,\bullet};P) \leq 1 < \frac{5}{3}$ or $S(W^{E}_{\bullet,\bullet};P) \leq \frac{3}{2} < \frac{5}{3}$. We get $\delta_P(S) = \frac{3}{5}$ for $P \in E$.

Step 2. Suppose $P \in E_1$. The Zariski decomposition of the divisor $-K_S - vE_1$ is:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{4}(6E + 3E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \big[0,1\big] \\ -K_S - vE_1 - \frac{v}{4}(6E + 3E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) - (v-1)A_1 \text{ if } v \in \big[1,2\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(6E + 3E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \big[0,1\big] \\ \frac{v}{4}(6E + 3E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) + (v-1)A_1 \text{ if } v \in \big[1,2\big] \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_1 \sim_{\mathbb{R}} (2-v)E_1 + \frac{1}{2}\Big(6E + 3E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7 + 2A_1\Big)$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{2} \text{ if } v \in \big[0,1\big] \\ \frac{(2-v)^2}{2} \text{ if } v \in \big[1,2\big] \end{cases} \quad \text{and } P(v)\cdot E_1 = \begin{cases} \frac{v}{2} \text{ if } v \in \big[0,1\big] \\ 1 - \frac{v}{2} \text{ if } v \in \big[1,2\big] \end{cases}$$

Now we apply the computation from Section 14 (Step 1.) and get $\delta_P(S) = 1$ for $P \in E_1 \backslash E$.

Step 3. Suppose $P \in E_2$. The Zariski decomposition of the divisor $-K_S - vE_2$ is the following:

$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{4}(6E + 3E_1 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \big[0,\frac{4}{3}\big] \\ -K_S - vE_2 - (v-1)(6E + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) - (6v-7)E_1 - (6v-8)A_1 \text{ if } v \in \big[\frac{4}{3},\frac{3}{2}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(6E + 3E_1 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \big[0,\frac{4}{3}\big] \\ (v-1)(6E + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7) + (6v-7)E_1 + (6v-8)A_1 \text{ if } v \in \big[\frac{4}{3},\frac{3}{2}\big] \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_2 \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)E_2 + \frac{1}{2}\Big(6E + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7 + 4E_1 + 2A_1\Big)$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{2} \text{ if } v \in \big[0,\frac{4}{3}\big] \\ (3-2v)^2 \text{ if } v \in \big[\frac{4}{3},\frac{3}{2}\big] \end{cases} \quad \text{and } P(v)\cdot E_2 = \begin{cases} \frac{v}{2} \text{ if } v \in \big[0,\frac{4}{3}\big] \\ 2(3-2v) \text{ if } v \in \big[\frac{4}{3},\frac{3}{2}\big] \end{cases}$$

We have $S_S(E_2) = \frac{17}{18}$. Thus, $\delta_P(S) \leq \frac{18}{17}$ for $P \in E_2$. Moreover, if $P \in E_2 \backslash E_1$ we have:

$$h(v) \leq \begin{cases} \frac{v^2}{8} \text{ if } v \in \big[0,\frac{4}{3}\big] \\ 2(3-2v)^2 \text{ if } v \in \big[\frac{4}{3},\frac{3}{2}\big] \end{cases}$$

Thus, $S(W^{E_2}_{\bullet,\bullet};P) \leq \frac{2}{9} < \frac{17}{18}$. We get $\delta_P(S) = \frac{18}{17}$ for $P \in E_2 \backslash E_1$.

Step 4. Suppose $P \in E_3$. The Zariski decomposition of the divisor $-K_S - vE_3$ is the following:

$$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{2}(2E + E_1 + E_2) - \frac{v}{5}(4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in \big[0,2\big] \\ -K_S - vE_3 - (v-1)(2E + E_2) - \frac{v}{5}(4E_3 + 3E_4 + 2E_5 + E_6) - (2v-3)E_1 - (2v-4)A_1 \text{ if } v \in \big[2,\frac{5}{2}\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(2E + E_1 + E_2) + \frac{v}{5}(4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in \big[0,2\big] \\ (v-1)(2E + E_2) + \frac{v}{5}(4E_3 + 3E_4 + 2E_5 + E_6) + (2v-3)E_1 + (2v-4)A_1 \text{ if } v \in \big[2,\frac{5}{2}\big] \end{cases}$$

The Zariski Decomposition follows from

$$-K_S - vE_3 \sim_{\mathbb{R}} \Big(\frac{5}{2} - v\Big)E_3 + \frac{1}{2}\Big(6E + 3E_2 + 4E_3 + 3E_4 + 2E_5 + E_6 + 4E_1 + 2A_1\Big)$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{5} \text{ if } v \in \big[0, 2\big] \\ \frac{(5-2v)^2}{5} \text{ if } v \in \big[2, \frac{5}{2}\big] \end{cases} \quad \text{and } P(v)\cdot E_3 = \begin{cases} \frac{v}{5} \text{ if } v \in \big[0, 2\big] \\ 2(1 - \frac{2v}{5}) \text{ if } v \in \big[2, \frac{5}{2}\big] \end{cases}$$

Now we apply the computation from Section 17 (Step 1.) and get that $\delta_P(S) = \frac{2}{3}$ for $P \in E_3\backslash E$.

Step 5. Suppose $P \in E_4$. The Zariski decomposition of the divisor $-K_S - vE_4 \sim C + E_1 + E_2 + 2E + 2E_3 + (2-v)E_4 + 2E_5 + 2E_6 + E_7$ is given by:

$$P(v) = -K_S - vE_4 - \frac{v}{4}(2E_1 + 2E_2 + 4E + 4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in \big[0, 2\big]$$
$$N(v) = \frac{v}{4}(2E_1 + 2E_2 + 4E + 4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in \big[0, 2\big]$$

Moreover,

$$(P(v))^2 = \frac{(2-v)(2+v)}{4} \text{ and } P(v)\cdot E_4 = \frac{v}{4} \text{ if } v \in \big[0, 2\big]$$

Now we apply the computation from Section 16 (Step 1.) and get that $\delta_P(S) = \frac{3}{4}$ for $P \in E_4\backslash E_3$.

Step 6. Suppose $P \in E_5$. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities where the configuration of all the (-1) and (-2) curves is known so the Zariski decomposition of the divisor $-K_S - vE_5 \sim C + E_1 + E_2 + 2E + 2E_3 + 2E_4 + (2-v)E_5 + 2E_6 + E_7$ is:

$$P(v) = \begin{cases} -K_S - vE_5 - \frac{v}{6}(3E_1 + 3E_2 + 6E + 6E_3 + 6E_4 + 4E_6 + 2E_7) \text{ if } v \in \big[0, \frac{3}{2}\big] \\ -K_S - vE_5 - \frac{v}{2}(E_1 + E_2 + 2E + 2E_3 + 2E_4) - (v-1)(2E_6 + E_7) - (2v-3)C \text{ if } v \in \big[\frac{3}{2}, 2\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{6}(3E_1 + 3E_2 + 6E + 6E_3 + 6E_4 + 4E_6 + 2E_7) \text{ if } v \in \big[0, \frac{3}{2}\big] \\ \frac{v}{2}(E_1 + E_2 + 2E + 2E_3 + 2E_4) + (v-1)(2E_6 + E_7) + (2v-3)C \text{ if } v \in \big[\frac{3}{2}, 2\big] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{3} \text{ if } v \in \big[0, \frac{3}{2}\big] \\ (2-v)^2 \text{ if } v \in \big[\frac{3}{2}, 2\big] \end{cases} \quad \text{and } P(v)\cdot E_5 = \begin{cases} \frac{v}{3} \text{ if } v \in \big[0, \frac{3}{2}\big] \\ 2 - v \text{ if } v \in \big[\frac{3}{2}, 2\big] \end{cases}$$

Now we apply the computation from Section 15 (Step 1.) and get that $\delta_P(S) = \frac{6}{7}$ for $P \in E_5\backslash E_4$.

Step 7. Suppose $P \in E_6$. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities where the configuration of all the (-1) and (-2) curves is known so the Zariski decomposition of the divisor $-K_S - vE_6 \sim C + E_1 + E_2 + 2E + 2E_3 + 2E_4 + 2E_5 + (2-v)E_6 + E_7$ is:

$$P(v) = \begin{cases} -K_S - vE_6 - \frac{v}{2}(2E_5 + 2E_4 + 2E_3 + 2E + E_1 + E_2 + E_6) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE_6 - \frac{v}{2}(2E_5 + 2E_4 + 2E_3 + 2E + E_1 + E_2 + E_6) - (v-1)C \text{ if } v \in \big[1, 2\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(2E_5 + 2E_4 + 2E_3 + 2E + E_1 + E_2 + E_6) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{2}(2E_5 + 2E_4 + 2E_3 + 2E + E_1 + E_2 + E_6) + (v-1)C \text{ if } v \in \big[1, 2\big] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{2} \text{ if } v \in \big[0, 1\big] \\ \frac{(2-v)^2}{2} \text{ if } v \in \big[1, 2\big] \end{cases} \quad \text{and } P(v)\cdot E_6 = \begin{cases} \frac{v}{2} \text{ if } v \in \big[0, 1\big] \\ 1 - \frac{v}{2} \text{ if } v \in \big[1, 2\big] \end{cases}$$

Now we apply the computation from Section 14 (Step 1.) and get that $\delta_P(S)=1$ for $P\in E_6\backslash E_5$.
Step 8. Suppose $P\in E_7$. The Zariski decomposition of the divisor $-K_S-vE_7\sim C+E_1+E_2+2E+2E_3+2E_4+2E_5+2E_6+(1-v)E_7$ is given by:

$$P(v)=-K_S-vE_7-\frac{v}{2}(2E_6+2E_5+2E_4+2E_3+2E+E_1+E_2) \text{ if } v\in\big[0,1\big]$$
$$N(v)=\frac{v}{2}(2E_6+2E_5+2E_4+2E_3+2E+E_1+E_2) \text{ if } v\in\big[0,1\big]$$

Moreover,

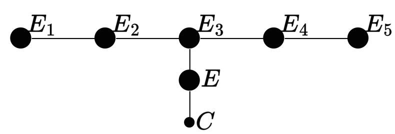
$$(P(v))^2=(1-v)(1+v) \text{ and } P(v)\cdot E_7=v \text{ if } v\in\big[0,1\big]$$

Now we apply the computation from Section 14 (Step 2.) and get that $\delta_P(S)=\frac{3}{2}$ for $P\in E_7\backslash E_6$. Thus, $\delta_{\mathcal{P}}(X)=\frac{3}{5}$. □

19. $\mathbb{E}_6$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{E}_6$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X)=\frac{3}{5}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E, E_1, E_2, E_3, E_4 and E_5 are the exceptional divisors with the intersection:



We have $-K_S\sim C+E_1+2E_2+3E_3+2E_4+E_5+2E$. Let P be a point on S.
Step 1. Suppose $P\in E_3$. The Zariski decomposition of the divisor $-K_S-vE_3\sim C+E_1+2E_2+(3-v)E_3+2E_4+E_5+2E$ is the following:

$$P(v)=\begin{cases}-K_S-vE_3-\frac{v}{3}(E_1+2E_2+2E_4+E_5)-\frac{v}{2}E \text{ if } v\in\big[0,2\big]\\ -K_S-vE_3-\frac{v}{3}(E_1+2E_2+2E_4+E_5)-(v-1)E-(v-2)C \text{ if } v\in\big[2,3\big]\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{3}(E_1+2E_2+2E_4+E_5)+\frac{v}{2}E \text{ if } v\in\big[0,2\big]\\ \frac{v}{3}(E_1+2E_2+2E_4+E_5)+(v-1)E+(v-2)C \text{ if } v\in\big[2,3\big]\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{v^2}{6} \text{ if } v\in\big[0,2\big]\\ \frac{(3-v)^2}{3} \text{ if } v\in\big[2,3\big]\end{cases} \quad\text{and } P(v)\cdot E_3=\begin{cases}\frac{v}{6} \text{ if } v\in\big[0,2\big]\\ 1-\frac{v}{6} \text{ if } v\in\big[2,3\big]\end{cases}$$

Now we apply the computation from Section 18 (Step 1.) and get that $\delta_P(S)=\frac{3}{5}$ for $P\in E_3$.
Step 2. Suppose $P\in E_2\cup E_4$. Without loss of generality we can assume that $P\in E_2$ since the proof is similar in other cases. The Zariski decomposition of the divisor $-K_S-vE_2\sim C+E_1+(2-v)E_2+3E_3+2E_4+E_5+2$ is:

$$P(v)=\begin{cases}-K_S-vE_2-\frac{v}{2}E_1-\frac{v}{5}(3E+6E_3+4E_4+2E_5) \text{ if } v\in\big[0,\frac{5}{3}\big]\\ -K_S-vE_2-\frac{v}{2}E_1-(v-1)(3E_3+2E_4+E_5)-(3v-4)E-(3v-5)C \text{ if } v\in\big[\frac{5}{3},2\big]\end{cases}$$
$$N(v)=\begin{cases}\frac{v}{2}E_1+\frac{v}{5}(3E+6E_3+4E_4+2E_5) \text{ if } v\in\big[0,\frac{5}{3}\big]\\ \frac{v}{2}E_1+(v-1)(3E_3+2E_4+E_5)+(3v-4)E+(3v-5)C \text{ if } v\in\big[\frac{5}{3},2\big]\end{cases}$$

Moreover,

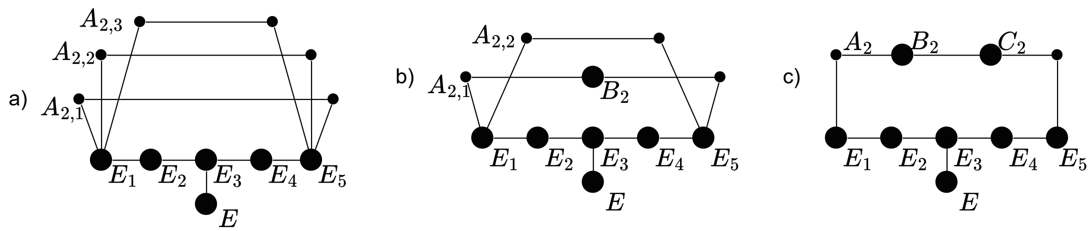
$$(P(v))^2 = \begin{cases} 1 - \frac{3v^2}{10} \text{ if } v \in \left[0, \frac{5}{3}\right] \\ \frac{3(2-v)^2}{2} \text{ if } v \in \left[\frac{5}{3}, 2\right] \end{cases} \quad \text{and } P(v) \cdot E_2 = \begin{cases} \frac{3v}{10} \text{ if } v \in \left[0, \frac{5}{3}\right] \\ 3(1 - \frac{v}{2}) \text{ if } v \in \left[\frac{5}{3}, 2\right] \end{cases}$$

We have $S_S(E_2) = \frac{11}{9}$. Thus, $\delta_P(S) \le \frac{9}{11}$ for $P \in E_2$. Moreover, if $P \in E_2 \backslash E_3$ we have

$$h(v) \le \begin{cases} \frac{39v^2}{200} \text{ if } v \in \left[0, \frac{5}{3}\right] \\ \frac{3(v-2)(v-6)}{8} \text{ if } v \in \left[\frac{5}{3}, 2\right] \end{cases}$$

Thus, $S(W^{E_2}_{\bullet,\bullet}; P) \le \frac{7}{9} < \frac{11}{9}$. We get $\delta_P(S) = \frac{9}{11}$ for $P \in (E_2 \cup E_4) \backslash E_3$.

Step 3. Suppose $P \in E_1 \cup E_5$. Without loss of generality we can assume that $P \in E_1$ since the proof is similar in other cases. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:



Then the corresponding Zariski Decomposition of the divisor $-K_S - vE_1$ is:

a).
$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{4}(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E) \text{ if } v \in \left[0, 1\right] \\ -K_S - vE_1 - \frac{v}{4}(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E) - (v-1)(A_{2,1} + A_{2,2} + A_{2,3}) \text{ if } v \in \left[1, \frac{4}{3}\right] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E) \text{ if } v \in \left[0, 1\right] \\ \frac{v}{4}(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E) + (v-1)(A_{2,1} + A_{2,2} + A_{2,3}) \text{ if } v \in \left[1, \frac{4}{3}\right] \end{cases}$$

b).
$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{4}(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E) \text{ if } v \in \left[0, 1\right] \\ -K_S - vE_1 - \frac{v}{4}(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E) - (v-1)(2A_{2,1} + B_{2,1} + A_{2,2}) \text{ if } v \in \left[1, \frac{4}{3}\right] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E) \text{ if } v \in \left[0, 1\right] \\ \frac{v}{4}(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E) + (v-1)(2A_{2,1} + B_{2,1} + A_{2,2}) \text{ if } v \in \left[1, \frac{4}{3}\right] \end{cases}$$

c).
$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{4}(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E) \text{ if } v \in \left[0, 1\right] \\ -K_S - vE_1 - \frac{v}{4}(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E) - (v-1)(3A_2 + B_2 + C_2) \text{ if } v \in \left[1, \frac{4}{3}\right] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E) \text{ if } v \in \left[0, 1\right] \\ \frac{v}{4}(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E) + (v-1)(3A_2 + B_2 + C_2) \text{ if } v \in \left[1, \frac{4}{3}\right] \end{cases}$$

The Zariski Decomposition in part a). follows from

$$-K_S - vE_1 \sim_{\mathbb{R}} \left(\frac{4}{3} - v\right)E_1 + \frac{1}{3}\left(5E_2 + 6E_3 + 4E_4 + 2E_5 + 3E + A_{2,1} + A_{2,2} + A_{2,3}\right)$$

A similar statement holds in other parts. Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{3v^2}{4} \text{ if } v \in \left[0, 1\right] \\ \frac{(4-3v)^2}{4} \text{ if } v \in \left[1, \frac{4}{3}\right] \end{cases} \quad \text{and } P(v) \cdot E_1 = \begin{cases} \frac{3v}{4} \text{ if } v \in \left[0, 1\right] \\ 3(1 - \frac{3v}{4}) \text{ if } v \in \left[1, \frac{4}{3}\right] \end{cases}$$

We apply the computation from Section 9 (Step 2.) and get $\delta_P(S) = \frac{3}{5}$ if $P \in (E_1 \cup E_5)\backslash(E_2 \cup E_4)$.
Step 4. Suppose $P \in E$. The Zariski decomposition of the divisor $-K_S - vE \sim C + E_1 + 2E_2 + 3E_3 + 2E_4 + E_5 + (2-v)E$ is:

$$P(v) = \begin{cases} -K_S - vE - \frac{v}{2}(E_1 + 2E_2 + 3E_3 + 2E_4 + E_5) \text{ if } v \in \big[0, 1\big] \\ -K_S - vE - \frac{v}{2}(E_1 + 2E_2 + 3E_3 + 2E_4 + E_5) - (v-1)C \text{ if } v \in \big[1, 2\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(E_1 + 2E_2 + 3E_3 + 2E_4 + E_5) \text{ if } v \in \big[0, 1\big] \\ \frac{v}{2}(E_1 + 2E_2 + 3E_3 + 2E_4 + E_5) + (v-1)C \text{ if } v \in \big[1, 2\big] \end{cases}$$

Moreover,

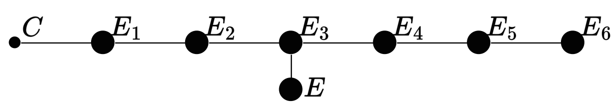
$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{2} \text{ if } v \in \big[0, 1\big] \\ \frac{(2-v)^2}{2} \text{ if } v \in \big[1, 2\big] \end{cases} \quad \text{and } P(v) \cdot E = \begin{cases} \frac{v}{2} \text{ if } v \in \big[0, 1\big] \\ 1 - \frac{v}{2} \text{ if } v \in \big[1, 2\big] \end{cases}$$

Now we apply the computation from Section 14 (Step 1.) and get that $\delta_P(S) = 1$ for $P \in E\backslash E_3$. Thus, $\delta_{\mathcal{P}}(X) = \frac{3}{5}$. □

20. $\mathbb{E}_7$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{E}_7$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X) = \frac{3}{7}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E, E_1, E_2, E_3, E_4, E_5 and E_6 are the exceptional divisors with the intersection:



We have $-K_S \sim C + 2E_1 + 3E_2 + 4E_3 + 3E_4 + 2E_5 + E_6 + 2E$. Let P be a point on S.
Step 1. Suppose $P \in E_3$. The Zariski decomposition of the divisor $-K_S - vE_3 \sim C + 2E_1 + 3E_2 + (4-v)E_3 + 3E_4 + 2E_5 + E_6 + 2E$ is the following:

$$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{4}(2E + 3E_4 + 2E_5 + E_6) - \frac{v}{3}(E_1 + 2E_2) \text{ if } v \in \big[0, 3\big] \\ -K_S - vE_3 - \frac{v}{4}(2E + 3E_4 + 2E_5 + E_6) - (v-1)E_1 - (v-2)E_2 - (v-3)C \text{ if } v \in \big[3, 4\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(2E + 3E_4 + 2E_5 + E_6) - \frac{v}{3}(E_1 + 2E_2) \text{ if } v \in \big[0, 3\big] \\ \frac{v}{4}(2E + 3E_4 + 2E_5 + E_6) + (v-1)E_1 + (v-2)E_2 + (v-3)C \text{ if } v \in \big[3, 4\big] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{12} \text{ if } v \in \big[0, 3\big] \\ \frac{(4-v)^2}{4} \text{ if } v \in \big[3, 4\big] \end{cases} \quad \text{and } P(v) \cdot E_3 = \begin{cases} \frac{v}{12} \text{ if } v \in \big[0, 3\big] \\ 1 - \frac{v}{4} \text{ if } v \in \big[3, 4\big] \end{cases}$$

We have $S_S(E_3) = \frac{7}{3}$. Thus, $\delta_P(S) \le \frac{3}{7}$ for $P \in E_3$. Moreover, if $P \in E_3 \cap (E \cup E_4)$ if $P \in E_3\backslash(E \cup E_4)$ we have

$$h(v) \le \begin{cases} \frac{19v^2}{228} \text{ if } v \in \big[0, 3\big] \\ \frac{(4-v)(5v+4)}{32} \text{ if } v \in \big[3, 4\big] \end{cases} \quad \text{or } h(v) \le \begin{cases} \frac{17v^2}{228} \text{ if } v \in \big[0, 3\big] \\ \frac{(4-v)(7v-4)}{32} \text{ if } v \in \big[3, 4\big] \end{cases}$$

Thus, $S(W^{E_3}_{\bullet,\bullet};P) \le \frac{11}{6} < \frac{7}{3}$ or $S(W^{E_3}_{\bullet,\bullet};P) \le \frac{5}{3} < \frac{7}{3}$. We get $\delta_P(S) = \frac{3}{7}$ for $P \in E_3$.

Step 2. Suppose $P \in E_2$. The Zariski decomposition of the divisor $-K_S - vE_2 \sim C + 2E_1 + (3-v)E_2 + 4E_3 + 3E_4 + 2E_5 + E_6 + 2E$ is the following:

$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{3}(2E + 4E_3 + 3E_4 + 2E_5 + E_6) - \frac{v}{2}E_1 \text{ if } v \in \big[0,2\big] \\ -K_S - vE_2 - \frac{v}{3}(2E + 4E_3 + 3E_4 + 2E_5 + E_6) - (v-1)E_1 - (v-2)C \text{ if } v \in \big[2,3\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2E + 4E_3 + 3E_4 + 2E_5 + E_6) + \frac{v}{2}E_1 \text{ if } v \in \big[0,2\big] \\ \frac{v}{3}(2E + 4E_3 + 3E_4 + 2E_5 + E_6) + (v-1)E_1 + (v-2)C \text{ if } v \in \big[2,3\big] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{6} \text{ if } v \in \big[0,2\big] \\ \frac{(3-v)^2}{3} \text{ if } v \in \big[2,3\big] \end{cases} \quad \text{and } P(v)\cdot E_2 = \begin{cases} \frac{v}{6} \text{ if } v \in \big[0,2\big] \\ 1 - \frac{v}{6} \text{ if } v \in \big[2,3\big] \end{cases}$$

Now we apply the computation from Section 18 (Step 1.) and get that $\delta_P(S) = \frac{3}{5}$ for $P \in E_2 \backslash E_3$.

Step 3. Suppose $P \in E_1$. The Zariski decomposition of the divisor $-K_S - vE_1 \sim C + (2-v)E_1 + 3E_2 + 4E_3 + 3E_4 + 2E_5 + E_6 + 2E$ is:

$$P(v) = \begin{cases} -K_S - vE_1 - \frac{v}{2}(2E + 3E_2 + 4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in \big[0,1\big] \\ -K_S - vE_1 - \frac{v}{2}(2E + 3E_2 + 4E_3 + 3E_4 + 2E_5 + E_6) - (v-1)C \text{ if } v \in \big[1,2\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(2E + 3E_2 + 4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in \big[0,1\big] \\ \frac{v}{2}(2E + 3E_2 + 4E_3 + 3E_4 + 2E_5 + E_6) + (v-1)C \text{ if } v \in \big[1,2\big] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{2} \text{ if } v \in \big[0,1\big] \\ \frac{(2-v)^2}{2} \text{ if } v \in \big[1,2\big] \end{cases} \quad \text{and } P(v)\cdot E_1 = \begin{cases} \frac{v}{2} \text{ if } v \in \big[0,1\big] \\ 1 - \frac{v}{2} \text{ if } v \in \big[1,2\big] \end{cases}$$

Now we apply the computation from Section 14 (Step 1.) and get that $\delta_P(S) = 1$ for $P \in E_1 \backslash E_2$.

Step 4. Suppose $P \in E$. The Zariski decomposition of the divisor $-K_S - vE \sim C + 2E_1 + 3E_2 + 4E_3 + 3E_4 + 2E_5 + E_6 + (2-v)E$ is:

$$P(v) = \begin{cases} -K_S - vE - \frac{v}{7}(4E_1 + 8E_2 + 12E_3 + 9E_4 + 6E_5 + 3E_6) \text{ if } v \in \big[0,\frac{7}{4}\big] \\ -K_S - vE - (4v-7)C - (4v-6)E_1 - (4v-5)E_2 - (v-1)(4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in \big[\frac{7}{4},2\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{7}(4E_1 + 8E_2 + 12E_3 + 9E_4 + 6E_5 + 3E_6) \text{ if } v \in \big[0,\frac{7}{4}\big] \\ (4v-7)C + (4v-6)E_1 + (4v-5)E_2 + (v-1)(4E_3 + 3E_4 + 2E_5 + E_6) \text{ if } v \in \big[\frac{7}{4},2\big] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{2v^2}{7} \text{ if } v \in \big[0,\frac{7}{4}\big] \\ 2(2-v)^2 \text{ if } v \in \big[\frac{7}{4},2\big] \end{cases} \quad \text{and } P(v)\cdot E = \begin{cases} \frac{2v}{7} \text{ if } v \in \big[0,\frac{7}{4}\big] \\ 2(2-v) \text{ if } v \in \big[\frac{7}{4},2\big] \end{cases}$$

We have $S_S(E) = \frac{5}{4}$. Thus, $\delta_P(S) \le \frac{4}{5}$ for $P \in E$. Moreover, if $P \in E \backslash E_3$ we have

$$h(v) \le \begin{cases} \frac{2v^2}{49} \text{ if } v \in \big[0,\frac{7}{4}\big] \\ 2(v-2)^2 \text{ if } v \in \big[\frac{7}{4},2\big] \end{cases}$$

Thus $S(W^{E_2}_{\bullet,\bullet};P) \le \frac{1}{6} < \frac{5}{4}$. We get $\delta_P(S) = \frac{4}{5}$ for $P \in E \backslash E_3$.

Step 5. Suppose $P \in E_4$. The Zariski decomposition of the divisor $-K_S - vE_4 \sim C + 2E_1 + 3E_2 +$

$4E_3+(3-v)E_4+2E_5+E_6+2E$ is:

$$P(v)=\begin{cases}-K_S-vE_4-\frac{v}{5}(2E_1+4E_2+6E_3+3E)-\frac{v}{3}(2E_5+E_6) \text{ if } v\in\left[0,\frac{5}{2}\right]\\ -K_S-vE_4-(2v-5)C-(2v-4)E_1-(2v-3)E_2-(2v-2)E_3-(v-1)E-\frac{v}{3}(2E_5+E_6) \text{ if } v\in\left[\frac{5}{2},3\right]\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{5}(2E_1+4E_2+6E_3+3E)-\frac{v}{3}(2E_5+E_6) \text{ if } v\in\left[0,\frac{5}{2}\right]\\ (2v-5)C+(2v-4)E_1+(2v-3)E_2+(2v-2)E_3+(v-1)E+\frac{v}{3}(2E_5+E_6) \text{ if } v\in\left[\frac{5}{2},3\right]\end{cases}$$

Moreover,

$$(P(v))^2=\begin{cases}1-\frac{2v^2}{15} \text{ if } v\in\left[0,\frac{5}{2}\right]\\ \frac{2(3-v)^2}{3} \text{ if } v\in\left[\frac{5}{2},3\right]\end{cases} \quad \text{and } P(v)\cdot E_4=\begin{cases}\frac{2v}{15} \text{ if } v\in\left[0,\frac{5}{2}\right]\\ 2(1-\frac{v}{3}) \text{ if } v\in\left[\frac{5}{2},3\right]\end{cases}$$

We have $S_S(E_4)=\frac{11}{6}$. Thus, $\delta_P(S)\le\frac{6}{11}$ for $P\in E_4$. Moreover, if $P\in E_4\backslash E_3$ we have

$$h(v)\le\begin{cases}\frac{22v^2}{225} \text{ if } v\in\left[0,\frac{5}{2}\right]\\ \frac{2(3-v)(v+3)}{9} \text{ if } v\in\left[\frac{5}{2},3\right]\end{cases}$$

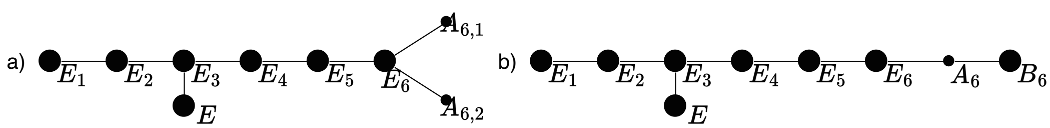
Thus, $S(W^{E_4}_{\bullet,\bullet};P)\le\frac{4}{3}<\frac{11}{6}$. We get $\delta_P(S)=\frac{6}{11}$ for $P\in E_4\backslash E_3$.
Step 6. Suppose $P\in E_5$. The Zariski decomposition of the divisor $-K_S-vE_5\sim C+2E_1+3E_2+4E_3+3E_4+(2-v)E_5+E_6+2E$ is given by:

$$P(v)=-K_S-vE_5-\frac{v}{4}(2E_1+4E_2+6E_3+5E_4+3E+2E_6) \text{ if } v\in\left[0,2\right]$$
$$N(v)=\frac{v}{4}(2E_1+4E_2+6E_3+5E_4+3E+2E_6) \text{ if } v\in\left[0,2\right]$$

Moreover,

$$(P(v))^2=\frac{(2-v)(2+v)}{4} \text{ and } P(v)\cdot E_5=\frac{v}{4} \text{ if } v\in\left[0,2\right]$$

Now we apply the computation from Section 16 (Step 1.) and get that $\delta_P(S)=\frac{3}{4}$ for $P\in E_5\backslash E_4$.
Step 7. Suppose $P\in E_6$. If we contract the curve C the resulting surface is isomorphic to a weak del Pezzo surface of degree 2 with at most Du Val singularities. Thus, there exist (-1)-curves and (-2)-curves which form one of the following dual graphs:



Then the corresponding Zariski Decomposition of the divisor $-K_S-vE_1$ is:

a). $$P(v)=\begin{cases}-K_S-vE_6-\frac{v}{3}(2E_1+4E_2+6E_3+5E_4+4E_5+3E) \text{ if } v\in\left[0,1\right]\\ -K_S-vE_6-\frac{v}{3}(2E_1+4E_2+6E_3+5E_4+4E_5+3E)-(v-1)(A_{6,1}+A_{6,2}) \text{ if } v\in\left[1,\frac{3}{2}\right]\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{3}(2E_1+4E_2+6E_3+5E_4+4E_5+3E) \text{ if } v\in\left[0,1\right]\\ \frac{v}{3}(2E_1+4E_2+6E_3+5E_4+4E_5+3E)+(v-1)(A_{1,1}+A_{1,2}) \text{ if } v\in\left[1,\frac{3}{2}\right]\end{cases}$$

b). $$P(v)=\begin{cases}-K_S-vE_6-\frac{v}{3}(2E_1+4E_2+6E_3+5E_4+4E_5+3E) \text{ if } v\in\left[0,1\right]\\ -K_S-vE_6-\frac{v}{3}(2E_1+4E_2+6E_3+5E_4+4E_5+3E)-(v-1)(2A_6+B_6) \text{ if } v\in\left[1,\frac{3}{2}\right]\end{cases}$$

$$N(v)=\begin{cases}\frac{v}{3}(2E_1+4E_2+6E_3+5E_4+4E_5+3E) \text{ if } v\in\left[0,1\right]\\ \frac{v}{3}(2E_1+4E_2+6E_3+5E_4+4E_5+3E)+(v-1)(2A_6+B_6) \text{ if } v\in\left[1,\frac{3}{2}\right]\end{cases}$$

The Zariski Decomposition in part a). follows from

$$-K_S - vE_6 \sim_{\mathbb{R}} \Big(\frac{3}{2} - v\Big)E_6 + \frac{1}{2}\Big(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E + A_{6,1} + A_{6,2}\Big)$$

A similar statement holds in other parts. Moreover,

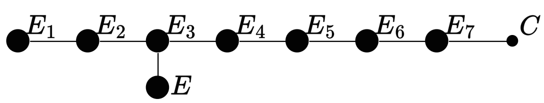
$$(P(v))^2 = \begin{cases} 1 - \frac{2v^2}{3} \text{ if } v \in \big[0,1\big] \\ \frac{(3-2v)^2}{3} \text{ if } v \in \big[1,\frac{3}{2}\big] \end{cases} \quad \text{and } P(v)\cdot E_1 = \begin{cases} \frac{2v}{3} \text{ if } v \in \big[0,1\big] \\ 2(1-\frac{2v}{3}) \text{ if } v \in \big[1,\frac{3}{2}\big] \end{cases}$$

Now we apply the computation from Section 16 (Step 2.) and get that $\delta_P(S) = \frac{6}{5}$ for $P \in E_6 \backslash E_5$. Thus, $\delta_{\mathcal{P}}(X) = \frac{3}{7}$. ☐

21. $\mathbb{E}_8$ singularity on Du Val Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 with an $\mathbb{E}_8$ singularity at point $\mathcal{P}$. Let $\mathcal{C}$ be a curve in the pencil $|-K_X|$ that contains $\mathcal{P}$. One has $\delta_{\mathcal{P}}(X) = \frac{3}{11}$.

Proof. Let S be the minimal resolution of singularities. Then S is a weak del Pezzo surface of degree 1. Suppose C is a strict transform of $\mathcal{C}$ on S and E, E_1, E_2, E_3, E_4, E_5, E_6 and E_7 are the exceptional divisors with the intersection:



We have $-K_S \sim C + 2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + 3E$.
Step 1. Suppose $P \in E_3$. The Zariski decomposition of the divisor $-K_S - vE_3 \sim C + 2E_1 + 4E_2 + (6-v)E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + 3E$ is the following:

$$P(v) = \begin{cases} -K_S - vE_3 - \frac{v}{2}E - \frac{v}{3}(E_1 + 2E_2) - \frac{v}{5}(4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \big[0,5\big] \\ -K_S - vE_3 - \frac{v}{2}E - \frac{v}{3}(E_1 + 2E_2) - (v-1)E_4 - (v-2)E_5 - (v-3)E_6 - (v-4)E_7 - (v-5)C \text{ if } v \in \big[5,6\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}E + \frac{v}{3}(E_1 + 2E_2) + \frac{v}{5}(4E_4 + 3E_5 + 2E_6 + E_7) \text{ if } v \in \big[0,5\big] \\ \frac{v}{2}E + \frac{v}{3}(E_1 + 2E_2) + (v-1)E_4 + (v-2)E_5 + (v-3)E_6 + (v-4)E_7 + (v-5)C \text{ if } v \in \big[5,6\big] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{30} \text{ if } v \in \big[0,5\big] \\ \frac{(6-v)^2}{6} \text{ if } v \in \big[5,6\big] \end{cases} \quad \text{and } P(v)\cdot E_3 = \begin{cases} \frac{v}{30} \text{ if } v \in \big[0,5\big] \\ 1 - \frac{v}{6} \text{ if } v \in \big[5,6\big] \end{cases}$$

We have $S_S(E_3) = \frac{11}{3}$. Thus, $\delta_P(S) \leq \frac{3}{11}$ for $P \in E_3$. Moreover, if $P \in E_3 \cap (E \cup E_2)$ if $P \in E_3 \backslash (E \cup E_2)$ we have

$$h(v) \leq \begin{cases} \frac{41v^2}{1800} \text{ if } v \in \big[0,5\big] \\ \frac{(6-v)(7v+6)}{72} \text{ if } v \in \big[5,6\big] \end{cases} \quad \text{or } h(v) \leq \begin{cases} \frac{49v^2}{1800} \text{ if } v \in \big[0,5\big] \\ \frac{(6-v)(11v-6)}{72} \text{ if } v \in \big[5,6\big] \end{cases}$$

Thus, $S(W^{E_3}_{\bullet,\bullet}; P) \leq \frac{5}{2} < \frac{11}{3}$ or $S(W^{E_3}_{\bullet,\bullet}; P) \leq 3 < \frac{11}{3}$. We get $\delta_P(S) = \frac{3}{11}$ for $P \in E_3$.
Step 2. Suppose $P \in E_2$. The Zariski decomposition of the divisor $-K_S - vE_2 \sim C + 2E_1 + (4-v)E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + 3E$ is:

$$P(v) = \begin{cases} -K_S - vE_2 - \frac{v}{2}E_1 - \frac{v}{7}(5E + 10E_3 + 8E_4 + 6E_5 + 4E_6 + 2E_7) \text{ if } v \in \big[0,\frac{7}{2}\big] \\ -K_S - vE_2 - \frac{v}{2}E_1 - (v-1)E - (2v-2)E_3 - (2v-3)E_4 - (2v-4)E_5 - (2v-5)E_6 - (2v-6)E_7 - (2v-7)C \text{ if } v \in \big[\frac{7}{2},4\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}E_1 + \frac{v}{7}(5E + 10E_3 + 8E_4 + 6E_5 + 4E_6 + 2E_7) \text{ if } v \in \big[0,\frac{7}{2}\big] \\ \frac{v}{2}E_1 + (v-1)E + (2v-2)E_3 + (2v-3)E_4 + (2v-4)E_5 + (2v-5)E_6 + (2v-6)E_7 + (2v-7)C \text{ if } v \in \big[\frac{7}{2},4\big] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{14} \text{ if } v \in \left[0, \frac{7}{2}\right] \\ \frac{(4-v)^2}{2} \text{ if } v \in \left[\frac{7}{2}, 4\right] \end{cases} \quad \text{and } P(v) \cdot E_2 = \begin{cases} \frac{v}{14} \text{ if } v \in \left[0, \frac{7}{2}\right] \\ 2 - \frac{v}{2} \text{ if } v \in \left[\frac{7}{2}, 4\right] \end{cases}$$

We have $S_S(E_2) = \frac{5}{2}$. Thus, $\delta_P(S) \leq \frac{2}{5}$ for $P \in E_2$. Moreover, if $P \in E_2 \backslash E_3$ we have

$$h(v) \leq \begin{cases} \frac{15v^2}{392} \text{ if } v \in \left[0, \frac{7}{2}\right] \\ \frac{(4-v)(4+v)}{8} \text{ if } v \in \left[\frac{7}{2}, 4\right] \end{cases}$$

Thus, $S(W^{E_2}_{\bullet,\bullet}; P) \leq \frac{4}{3} < \frac{5}{2}$. We get $\delta_P(S) = \frac{2}{5}$ for $P \in E_2 \backslash E_3$.

Step 3. Suppose $P \in E_1$. The Zariski decomposition of the divisor $-K_S - vE_1 \sim C + (2-v)E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + 3E$ is given by:

$$P(v) = -K_S - vE_1 - \frac{v}{4}(5E + 7E_2 + 10E_3 + 8E_4 + 6E_5 + 4E_6 + 2E_7) \text{ if } v \in \left[0, 2\right]$$
$$N(v) = \frac{v}{4}(5E + 7E_2 + 10E_3 + 8E_4 + 6E_5 + 4E_6 + 2E_7) \text{ if } v \in \left[0, 2\right]$$

Moreover,

$$(P(v))^2 = \frac{(2-v)(2+v)}{4} \text{ and } P(v) \cdot E_1 = \frac{v}{4} \text{ if } v \in \left[0, 2\right]$$

Now we apply the computation from Section 16 (Step 1.) and get that $\delta_P(S) = \frac{3}{4}$ for $P \in E_1 \backslash E_2$.

Step 4. Suppose $P \in E$. The Zariski decomposition of the divisor $-K_S - vE \sim C + 2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + (3-v)E$ is:

$$P(v) = \begin{cases} -K_S - vE - \frac{v}{8}(5E_1 + 10E_2 + 15E_3 + 12E_4 + 9E_5 + 6E_6 + 3E_7) \text{ if } v \in \left[0, \frac{8}{3}\right] \\ -K_S - vE - (v-1)(E_1 + 2E_2 + 3E_3) - (3v-4)E_4 - (3v-5)E_5 - (3v-6)E_6 - (3v-7)E_7 - (3v-8)C \text{ if } v \in \left[\frac{8}{3}, 3\right] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{8}(5E_1 + 10E_2 + 15E_3 + 12E_4 + 9E_5 + 6E_6 + 3E_7) \text{ if } v \in \left[0, \frac{8}{3}\right] \\ (v-1)(E_1 + 2E_2 + 3E_3) + (3v-4)E_4 + (3v-5)E_5 + (3v-6)E_6 + (3v-7)E_7 + (3v-8)C \text{ if } v \in \left[\frac{8}{3}, 3\right] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{8} \text{ if } v \in \left[0, \frac{8}{3}\right] \\ (3-v)^2 \text{ if } v \in \left[\frac{8}{3}, 3\right] \end{cases} \quad \text{and } P(v) \cdot E = \begin{cases} \frac{v}{8} \text{ if } v \in \left[0, \frac{8}{3}\right] \\ 3 - v \text{ if } v \in \left[\frac{8}{3}, 3\right] \end{cases}$$

We have $S_S(E) = \frac{17}{9}$. Thus, $\delta_P(S) \leq \frac{9}{17}$ for $P \in E$. Moreover, if $P \in E \backslash E_3$ we have

$$h(v) \leq \begin{cases} \frac{v^2}{128} \text{ if } v \in \left[0, \frac{8}{3}\right] \\ \frac{(3-v)^2}{2} \text{ if } v \in \left[\frac{8}{3}, 3\right] \end{cases}$$

Thus, $S(W^{E}_{\bullet,\bullet}; P) \leq \frac{1}{9} < \frac{17}{9}$. We get $\delta_P(S) = \frac{9}{17}$ for $P \in E \backslash E_3$.

Step 5. Suppose $P \in E_4$. The Zariski decomposition of the divisor $-K_S - vE_4 \sim C + 2E_1 + 4E_2 + 6E_3 + (5-v)E_4 + 4E_5 + 3E_6 + 2E_7 + 3E$ is the following:

$$P(v) = \begin{cases} -K_S - vE_4 - \frac{v}{5}(2E_1 + 4E_2 + 6E_3 + 3E) - \frac{v}{4}(3E_5 + 2E_6 + E_7) \text{ if } v \in \left[0, 4\right] \\ -K_S - vE_4 - \frac{v}{5}(2E_1 + 4E_2 + 6E_3 + 3E) - (v-1)E_5 - (v-2)E_6 - (v-3)E_7 - (v-4)C \text{ if } v \in \left[4, 5\right] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{5}(2E_1 + 4E_2 + 6E_3 + 3E) + \frac{v}{4}(3E_5 + 2E_6 + E_7) \text{ if } v \in \left[0, 4\right] \\ \frac{v}{5}(2E_1 + 4E_2 + 6E_3 + 3E) + (v-1)E_5 + (v-2)E_6 + (v-3)E_7 + (v-4)C \text{ if } v \in \left[4, 5\right] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{20} \text{ if } v \in \left[0, 4\right] \\ \frac{(5-v)^2}{5} \text{ if } v \in \left[4, 5\right] \end{cases} \quad \text{and } P(v) \cdot E_4 = \begin{cases} \frac{v}{20} \text{ if } v \in \left[0, 4\right] \\ 1 - \frac{v}{5} \text{ if } v \in \left[4, 5\right] \end{cases}$$

We have $S_S(E_4) = 3$. Thus, $\delta_P(S) \leq \frac{1}{3}$ for $P \in E_4$. Moreover, if $P \in E_4 \backslash E_3$ we have

$$h(v) \leq \begin{cases} \frac{31v^2}{800} \text{ if } v \in \big[0,4\big] \\ \frac{(5-v)(9v-5)}{50} \text{ if } v \in \big[4,5\big] \end{cases}$$

Thus, $S(W^{E_4}_{\bullet,\bullet}; P) \leq \frac{7}{3} < 3$. We get $\delta_P(S) = 3$ for $P \in E_4 \backslash E_3$.
Step 6. Suppose $P \in E_5$. The Zariski decomposition of the divisor $-K_S - vE_5 \sim C + 2E_1 + 4E_2 + 6E_3 + 5E_4 + (4-v)E_5 + 3E_6 + 2E_7 + 3E$ is the following:

$$P(v) = \begin{cases} -K_S - vE_5 - \frac{v}{4}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 3E) - \frac{v}{3}(2E_6 + E_7) \text{ if } v \in \big[0,3\big] \\ -K_S - vE_5 - \frac{v}{4}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 3E) - (v-1)E_6 - (v-2)E_7 - (v-3)C \text{ if } v \in \big[3,4\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{4}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 3E) - \frac{v}{3}(2E_6 + E_7) \text{ if } v \in \big[0,3\big] \\ \frac{v}{4}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 3E) + (v-1)E_6 + (v-2)E_7 + (v-3)C \text{ if } v \in \big[3,4\big] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{12} \text{ if } v \in \big[0,3\big] \\ \frac{(4-v)^2}{4} \text{ if } v \in \big[3,4\big] \end{cases} \quad \text{and } P(v) \cdot E_5 = \begin{cases} \frac{v}{12} \text{ if } v \in \big[0,3\big] \\ 1 - \frac{v}{4} \text{ if } v \in \big[3,4\big] \end{cases}$$

Now we apply the computation from Section 20 (Step 1.) and get that $\delta_P(S) = \frac{3}{7}$ for $P \in E_5 \backslash E_4$.
Step 7. Suppose $P \in E_6$. The Zariski decomposition of the divisor $-K_S - vE_6 \sim C + 2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + (3-v)E_6 + 2E_7 + 3E$ is the following:

$$P(v) = \begin{cases} -K_S - vE_6 - \frac{v}{3}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E) - \frac{v}{2}E_7 \text{ if } v \in \big[0,2\big] \\ -K_S - vE_6 - \frac{v}{3}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E) - (v-1)E_7 - (v-2)C \text{ if } v \in \big[2,3\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{3}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E) + \frac{v}{2}E_7 \text{ if } v \in \big[0,2\big] \\ \frac{v}{3}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E) + (v-1)E_7 + (v-2)C \text{ if } v \in \big[2,3\big] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{6} \text{ if } v \in \big[0,2\big] \\ \frac{(3-v)^2}{3} \text{ if } v \in \big[2,3\big] \end{cases} \quad \text{and } P(v) \cdot E_6 = \begin{cases} \frac{v}{6} \text{ if } v \in \big[0,2\big] \\ 1 - \frac{v}{6} \text{ if } v \in \big[2,3\big] \end{cases}$$

Now we apply the computation from Section 18 (Step 1.) and get that $\delta_P(S) = \frac{3}{5}$ for $P \in E_6 \backslash E_5$.
Step 8. Suppose $P \in E_7$. The Zariski decomposition of the divisor $-K_S - vE_7 \sim C + 2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + (2-v)E_7 + 3E$ is:

$$P(v) = \begin{cases} -K_S - vE_7 - \frac{v}{2}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 3E) \text{ if } v \in \big[0,1\big] \\ -K_S - vE_7 - \frac{v}{2}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 3E) - (v-1)C \text{ if } v \in \big[1,2\big] \end{cases}$$

$$N(v) = \begin{cases} \frac{v}{2}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 3E) \text{ if } v \in \big[0,1\big] \\ \frac{v}{2}(2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 3E) + (v-1)C \text{ if } v \in \big[1,2\big] \end{cases}$$

Moreover,

$$(P(v))^2 = \begin{cases} 1 - \frac{v^2}{2} \text{ if } v \in \big[0,1\big] \\ \frac{(2-v)^2}{2} \text{ if } v \in \big[1,2\big] \end{cases} \quad \text{and } P(v) \cdot E_7 = \begin{cases} \frac{v}{2} \text{ if } v \in \big[0,1\big] \\ 1 - \frac{v}{2} \text{ if } v \in \big[1,2\big] \end{cases}$$

Now we apply the computation from Section 14 (Step 1.) and get that $\delta_P(S) = 1$ for $P \in E_7 \backslash E_6$.
Thus, $\delta_{\mathcal{P}}(X) = \frac{3}{11}$. $\square$

Contents

References

[1] H. Abban, Z. Zhuang, *Seshadri constants and K-stability of Fano manifolds*, Duke Mathematical Journal, to appear.

[2] H. Abban, Z. Zhuang, *K-stability of Fano varieties via admissible flags*, Forum of Mathematics, Pi **10** (2022), Paper No. e15, 43 p.

[3] C. Araujo, A.-M. Castravet, I. Cheltsov, K. Fujita, A.-S. Kaloghiros, J. Martinez-Garcia, C. Shramov, H. Suss, and N. Viswanathan. *The Calabi problem for Fano threefolds.*, 2021

[4] G. Belousov, K. Loginov, *K-stability of Fano threefolds of rank 4 and degree 24.* European Journal of Mathematics 9, 80 (2023). https://doi.org/10.1007/s40879-023-00669-2

[5] G. Belousov, K. Loginov, *K-stability of Fano threefolds of rank 3 and degree 14.* Ann Univ Ferrara 70, 1093–1114 (2024). https://doi.org/10.1007/s11565-024-00526-4

[6] H. Blum, C. Xu, *Uniqueness of K-polystable degenerations of Fano varieties,* Ann. Math. 190 (2019), 609–656

[7] I. Cheltsov, *O*n singular cubic surfaces, Asian Journal of Mathematics 13 (2009), 191-214

[8] I. Cheltsov, D. Kosta, *C*omputing alpha-invariants of singular del Pezzo surfaces, Journal of Geometric Analysis 24 (2014), 798-842

[9] I. Cheltsov, E. Denisova, K. Fujita *K-stable smooth Fano threefolds of Picard rank two.* Forum of Mathematics, Sigma. 2024;12:e41. doi:10.1017/fms.2024.5

[10] I. Cheltsov, K. Fujita, T. Kishimoto, J. Park *K-stable Fano 3-folds in the families 2.18 and 3.4,* (2023) preprint arXiv:2304.11334

[11] I. Cheltsov, K. Fujita, T. Kishimoto, T. Okada, (2023). *K-Stable Divisors In* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ *of degree* $(1,1,2)$. Nagoya Mathematical Journal, 251, 686-714. https://doi.org/10.1017/nmj.2023.5

[12] I. Cheltsov, J. Park, *K-stable Fano threefolds of rank 2 and degree 30.* European Journal of Mathematics 8, 834–852 (2022). https://doi.org/10.1007/s40879-022-00569-x

[13] I. Cheltsov, Y. Prokhorov *Del Pezzo surfaces with infinite automorphism groups*, Algebraic Geometry 8 (2021), 319-357

[14] D. Coray, M. Tsfasman, *Arithmetic on singular Del Pezzo surfaces*, Proc. LMS **57** (1988), 25–87.

[15] E. Denisova, *δ-invariant of Du Val del Pezzo surfaces of degree* ≥ 4, preprint, arXiv:2304.11412, 2023.

[16] E. Denisova, *δ-invariant of cubic surfaces with Du Val singularities*, preprint, arXiv:2311.14181, 2023.

[17] E. Denisova, *δ-invariant of Du Val del Pezzo surfaces of degree* 2, preprint, arXiv: arXiv:2410.12512 (2024)

[18] E. Denisova, (2024). *On K-stability of P3 blown up along the disjoint union of a twisted cubic curve and a line.* Journal of the London Mathematical Society, 109(5), Article e12911. https://doi.org/10.1112/jlms.12911

[19] E. Denisova, *K-stability of Fano 3-folds of Picard rank 3 and degree 20.* Ann Univ Ferrara 70, 987–1022 (2024). https://doi.org/10.1007/s11565-024-00516-6

[20] T. D. Guerreiro, L. Giovenzana, N. Viswanathan *On K-stability of* $\mathbb{P}^3$ *blown up along a* $(2,3)$ *complete intersection*, (2023) Journal of the London Mathematical Society.

[21] K. Fujita, *A valuative criterion for uniform K-stability of* $\mathbb{Q}$*-Fano varieties*, J. Reine Angew. Math. **751** (2019), 309–338.

[22] V. Iskovskikh, *Fano 3-folds I*, Math. USSR, Izv. **11** (1977), 485–527.

[23] V. Iskovskikh, *Fano 3-folds II*, Math. USSR, Izv. **12** (1978), 469–506.

[24] C. Li, *K-semistability is equivariant volume minimization*, Duke Math. Jour. **166** (2017), 3147–3218.

[25] Y. Liu, *K-stability of Fano threefolds of rank 2 and degree 14 as double covers.* Math. Z. 303, 38 (2023). https://doi.org/10.1007/s00209-022-03192-4

[26] Y. Liu, J.Zhao *K-moduli of Fano threefolds and genus four curves*, preprint arXiv:2403.16747

[27] J. Malbon, *Automorphisms of Fano threefolds of rank 2 and degree 28.* Ann Univ Ferrara 70, 1083–1092 (2024). https://doi.org/10.1007/s11565-024-00525-5

[28] S. Mori, S. Mukai, *Classification of Fano* 3*-folds with* $B_2 \geqslant 2$, Manuscr. Math. **36** (1981), 147–162.

[29] S. Mori, S. Mukai, *Classification of Fano* 3*-folds with* $B_2 \geqslant 2$*. Erratum*, Manuscr. Math. **110** (2003), 407.

[30] Y. Odaka, C. Spotti, S. Sun, *Compact moduli spaces of del Pezzo surfaces and Kahler–Einstein metrics*, J. Differ. Geom. 102 (2016), 127–172.

[31] J. Park, J. Won, *L*og-canonical thresholds on Du Val Del Pezzo surfaces of degrees ≥ 2, Nagoya Math. J. 200, 1-26 (2010).

[32] J. Park, J. Won, *L*og canonical thresholds on Gorenstein canonical del Pezzo surfaces, Proc. Edinb. Math. Soc., II. Ser. 54, No. 1, 187-219 (2011).

[33] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math. **101** (1990), 101–172.

[34] G. Tian, S.-T. Yau, *Kähler–Einstein metrics on complex surfaces with* $C_1 > 0$, Commun. Math. Phys. **112**, (1987), 175–203.

[35] T. Urabe, *On singularities on degenerate Del Pezzo surfaces of degree 1, 2*, Proceedings of Symposia in Pure Mathematics, 40 Part 2 (1983), 587-590

[36] C. Xu, Y. Liu, *K-stability of cubic threefolds*, Duke Mathematical Journal **168** (2019), 2029–2073.

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